Projection onto a given divergence

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1 Abstract Problem

1.1 Definitions

Let H denote a Hilbert space with inner product denoted by $\langle,\rangle$ and norm $\|\cdot\|$ defined by $\|\mathbf{w}\| = \langle \mathbf{w}, \mathbf{w}\rangle^{1/2} \quad \forall \mathbf{w}$.

1.2 Problem

Given vectors $\mathbf{v}$ and $\mathbf{s}$, minimize $\|\mathbf{u}-\mathbf{v}\|^2$ subject to the constraint that $\operatorname{div}\mathbf{u} = \mathbf{s}$.

1.3 Solution framework

Suppose that $\mathbf{u}$ is the minimizer. Let $\mathbf{f} = \mathbf{u}+\mathbf{w}$ also satisfy $\operatorname{div}\mathbf{f} = \mathbf{s}$, i.e., $\operatorname{div}\mathbf{w} = 0$. Now $\|\mathbf{f}-\mathbf{v}\|^2 = \|(\mathbf{u}-\mathbf{v})+\mathbf{w}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 + 2\langle \mathbf{u}-\mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2$. Since we could replace $\mathbf{w}$ with its opposite, this is minimized at $\mathbf{w} = 0$ only if $\langle \mathbf{u}-\mathbf{v}, \mathbf{w}\rangle = 0$. (That is, $\mathbf{u}$ is the ***orthogonal* projection** of $\mathbf{v}$ onto the linear manifold of all $\mathbf{f}$ satisfying $\operatorname{div}\mathbf{f} = \mathbf{s}$.)

In general we will claim that the minimizer $\mathbf{u}$ is specified by

$$(\mathbf{u}-\mathbf{v}) = \operatorname{grad}\lambda, \tag{1}$$

where λ is restricted to belong to a class of functions satisfying the adjoint property

$$\langle \operatorname{grad}\lambda, \mathbf{w}\rangle = -\langle \lambda, \operatorname{div}\mathbf{w}\rangle. \tag{2}$$

In each particular case, we show that λ will satisfy this adjoint property if we require λ to satisfy an appropriate Dirichlet boundary condition of the form $\lambda = 0$ on $\partial\Omega$.

Since $\operatorname{div}\mathbf{w} = 0$, it is enough for there to exist a λ satisfying (8) and satisfying (7). Substituting (8) into the constraint $\operatorname{div}\mathbf{u} = 0$ gives the abstract Poisson equation

$$\operatorname{div}\operatorname{grad}\lambda = \mathbf{s} - \operatorname{div}\mathbf{v}. \tag{3}$$

So the problem reduces to showing that there is a unique λ that solves the Poisson equation (9) from a class of vectors λ which satisfy the adjoint property (7).

2 Continuum problem

2.1 Definitions of continuum problem

Let Ω be a nice domain.
For $\mathbf{u}, \mathbf{w}$ vector fields on Ω, let $\langle \mathbf{u}, \mathbf{w}\rangle := \int_\Omega \mathbf{u}\cdot\mathbf{w}$.

2.2 Statement of continuum problem

Let $\mathbf{v}$ be a vector field on the domain Ω.
Let σ be a desired divergence.
Find $\mathbf{u}$ that minimizes $\|\mathbf{u}-\mathbf{v}\|^2$ subject to the constraint that $\nabla\cdot\mathbf{u} = \sigma$ in Ω.

2.3 Solution

For this continuum problem there exists a unique solution to the Poisson equation (9) with Dirichlet boundary conditions $\lambda = 0$ on $\partial\Omega$, and such λ indeed satisfies the adjoint property (7):

$\langle \operatorname{grad}\lambda, \mathbf{w}\rangle = \int_\Omega (\nabla\lambda)\cdot\mathbf{w} = \int_\Omega \nabla\cdot(\lambda\mathbf{w}) - \int_\Omega \lambda\nabla\cdot\mathbf{w}$
$= -\langle \lambda, \nabla\cdot\mathbf{w}\rangle$, as needed.

3 Definitions for discrete calculus

Let $\langle \mathbf{f}, \mathbf{g}\rangle_a^b = \sum_{i=a}^b f_i g_i$ denote a generalized inner product.
Let $E^k = \mathbf{f} \mapsto \{f_{i+k}\}_{i\in\mathbb{Z}}$ be the shift operator. Let $E^+ := E^{+1}$ and $E^- := E^{-1}$.
Let $D^+ := E^+ - E^0$
Let $D^- := E^0 - E^-$

Observe that $\langle \mathbf{f}, \mathbf{g}\rangle_a^b = \langle E^k\mathbf{f}, E^k\mathbf{g}\rangle_{a-k}^{b-k}$.

4 Staggered discrete 1D problem

4.1 Problem

Given the scalar sequences $\mathbf{v} = \{v_i\}_{i=1}^m$ and $\mathbf{s} = \{s_i\}_{i=1}^{m-1}$, find $\mathbf{u} = \{u_i\}_{i=1}^m$ that minimizes $\|\mathbf{u}-\mathbf{v}\| = \sum_{i=1}^m (u_i - v_i)^2$, subject to the constraint that $(D^+\mathbf{u})_i = s_i$ for $i = 1, \ldots, (m-1)$.

4.2 Solution

Adopt the following definitions
Let $\operatorname{div} = D^+$.
Let $\operatorname{grad} = D^-$.
Require that λ satisfy the Dirichlet boundary conditions

$$\lambda_0 = 0 = \lambda_m. \tag{4}$$

We need to show that for such λ the following properties hold.

1. λ satisfies the adjoint property. Indeed,

   $\langle \operatorname{grad}\lambda, \mathbf{w}\rangle := \langle D^-\lambda, \mathbf{w}\rangle_1^m$
   $= \langle \lambda, \mathbf{w}\rangle_1^m - \langle E^-\lambda, \mathbf{w}\rangle_1^m$
   $= \langle \lambda, \mathbf{w}\rangle_1^m - \langle \lambda, E^+\mathbf{w}\rangle_0^{m-1}$
   $= -\langle \lambda, D^+\mathbf{w}\rangle_1^{m-1} + \lambda_m w_m - \lambda_0 w_1$
   $= 0$, using (4) and $\operatorname{div}\mathbf{w} = 0$.

2. There is a unique λ that satisfies the Poisson equation (9).

   To show this, we write out the Poisson equation (9) explictly as a linear system:

   $$-\lambda_{i+1} + 2\lambda_i - \lambda_{i-1} = g_i \text{ for } 1 \le i \le (m-1),$$
   $$\text{where } \mathbf{g} := \operatorname{div}\mathbf{v} - \mathbf{s}.$$

   Using the Dirichlet boundary conditions $\lambda_0 = 0 = \lambda_m$ and writing the system in matrix form, we see that we have a tridiagonal system:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m-2} \\ \lambda_{m-1} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{m-2} \\ g_{m-1} \end{bmatrix}$$

5 Staggered discretized 1D problems

5.1 Staggered divergence problem

Find $\mathbf{u}$ that minimizes $\|\mathbf{u}-\mathbf{v}\| := \sum_{i=1}^{m}(u_i - v_i)^2$ subject to $\frac{u_{i+1}-u_i}{dx} = s_{i+1/2}$ for $i = 1, \ldots, (m-1)$.

Solution: Let $\tilde{s}_i = (dx)\, s_{i+1/2}$, and map onto the previous problem. (It's also helpful to consider a mapping $\tilde{\lambda}_i = \lambda_{i+1/2}$.)

5.2 Staggered vector problem

Find $\mathbf{u}$ that minimizes $\|\mathbf{u}-\mathbf{v}\| := \sum_{i=1}^{n}(u_{i-1/2} - v_{i-1/2})^2$ subject to $\frac{u_{i+1/2}-u_{i-1/2}}{dx} = s_i$ for $1 \le i \le (n-1)$.

Solution: Let $\tilde{u}_i = u_{i-1/2}$, let $\tilde{\mathbf{s}} = (dx)\,\mathbf{s}$, let $m = n-1$, and map onto the previous problem.

A Even/odd discrete 1D problem

A.1 Problem

Find $\mathbf{u} = \{u_i\}_{i=0}^{m+1}$ that minimizes $\|\mathbf{u}-\mathbf{v}\| = \sum_{i=0}^{m+1}(u_i - v_i)^2$, subject to the constraint that $(D\mathbf{u})_i = s_i$ for $i = 1, \ldots, m$, where $D := E^+ - E^-$.

A.2 Solution

Let $n = m+1$.
Let $\operatorname{div} : \mathbf{u} \mapsto \{u_{i+1} - u_{i-1}\}_{i=1}^{m}$ be the discrete divergence operator.
Let $\operatorname{grad} : \lambda \mapsto \{\lambda_{i+1} - \lambda_{i-1}\}_{i=0}^{n}$ be the discrete gradient operator.
Let $\langle \mathbf{f}, \mathbf{g} \rangle := \langle \mathbf{f}, \mathbf{g} \rangle_0^n$ for $\mathbf{f}, \mathbf{g} \in V$ and let $\langle \lambda, \operatorname{div} \mathbf{f} \rangle := \langle \lambda, \operatorname{div} \mathbf{f} \rangle_1^m$ denote default inner products.

Impose the boundary conditions

$$0 = \lambda_{m+2} = \lambda_{m+1} \text{ and } 0 = \lambda_{-1} = \lambda_0, \tag{5}$$

For Section 1 to go through, we must verify the following two properties.

- We need that λ satisfies the adjoint property $\langle \operatorname{grad} \lambda, \mathbf{w} \rangle = 0$, as in (7). Indeed:

  $\langle \operatorname{grad} \lambda, \mathbf{w} \rangle := \langle \operatorname{grad} \lambda, \mathbf{w} \rangle_0^n$
  $= \langle E^+\lambda, \mathbf{w} \rangle_0^n - \langle E^-\lambda, \mathbf{w} \rangle_0^n$
  $= \langle \lambda, E^-\mathbf{w} \rangle_1^{n+1} - \langle \lambda, E^+\mathbf{w} \rangle_{-1}^{n-1}$
  $= \lambda_{n+1} w_n + \lambda_n w_{n-1} - \lambda_0 w_1 - \lambda_{-1} w_0 - \langle \lambda, \operatorname{div} \mathbf{w} \rangle_1^m$
  $= 0$, using (5) and $\operatorname{div} \mathbf{w} = 0$.

  This decouples the system into a pair of tridiagonal systems for even and odd indices.

- We need that

  $$\operatorname{div} \operatorname{grad} \lambda = \mathbf{s} - \operatorname{div} \mathbf{v}. \tag{6}$$

  as in (9).

  Writing out the system explicitly and using the Dirichlet boundary conditions gives a decoupled pair of tridiagonal systems for even and odd indices.

B Even/odd discretized 1-D problem

Let dx be the mesh size.
Let $\operatorname{div} : \mathbf{u} \mapsto \{\frac{u_{i+1}-u_{i-1}}{2\,dx}\}_{i=1}^{m}$ be the discrete divergence operator.
Let $\mathbf{s} = \{s_i\}_{i=1}^{m}$ be a desired discrete divergence.
Let $\operatorname{grad} : \lambda \mapsto \{\frac{\lambda_{i+1}-\lambda_{i-1}}{2\,dx}\}_{i=0}^{n}$ be the discrete gradient operator.

B.1 Problem

Find $\mathbf{u} = \{u_i\}_{i=0}^{n}$ that minimizes $\|\mathbf{u}-\mathbf{v}\| = \sum_{i=0}^{n}(u_i - v_i)^2$. subject to the constraint that $(\operatorname{div} \mathbf{u})_i = s_i$ for $i = 1, \ldots, m$.

Solution: Let $\tilde{\mathbf{s}} = 2\,dx\,\mathbf{s}$ and map onto the previous problem.

C Derivation of solution using Lagrange multipliers

C.1 Statement of abstract problem

We recall the abstract problem: Given vectors $\mathbf{v}$ and $\mathbf{s}$, minimize $\|\mathbf{u}-\mathbf{v}\|^2$ subject to the constraint that $\operatorname{div}\mathbf{u} = \mathbf{s}$.

C.2 Derivation of solution

Use Lagrange multipliers.
Let $L(\mathbf{u},\lambda) = \|\mathbf{u}-\mathbf{v}\|^2 + 2\langle\lambda, \operatorname{div}\mathbf{u}-\mathbf{s}\rangle$. We seek a stationary point.

Set $0 = d_\epsilon|_{\epsilon=0} L(\mathbf{u},\lambda+\epsilon\lambda') = 2\langle\lambda', \operatorname{div}\mathbf{u}-\mathbf{s}\rangle$. Since this must be true for arbitrary λ', we recover the constraint equation $\operatorname{div}\mathbf{u} = \mathbf{s}$.

Set $0 = d_\epsilon|_{\epsilon=0} L(\mathbf{u}+\epsilon\mathbf{u}',\lambda)$
$= d_\epsilon|_{\epsilon=0}\Big(\|\mathbf{u}+\epsilon\mathbf{u}'-\mathbf{v}\|^2 + 2\langle\lambda, \operatorname{div}(\mathbf{u}+\epsilon\mathbf{u}')-\mathbf{s}\rangle\Big)$
$= d_\epsilon|_{\epsilon=0}\Big(\|\mathbf{u}-\mathbf{v}\|^2 + \epsilon 2\langle\mathbf{u}',\mathbf{u}-\mathbf{v}\rangle + \epsilon^2\|\mathbf{u}'\|^2\Big) + 2\langle\lambda, \operatorname{div}\mathbf{u}'\rangle$
$= 2\langle\mathbf{u}',\mathbf{u}-\mathbf{v}\rangle + 2\langle\lambda, \operatorname{div}\mathbf{u}'\rangle$.

At this point in the derivation one restricts $\mathbf{u}'$ by requiring $\mathbf{u}'$ to satisfy some appropriate kind of zero-value boundary condition so that.

$$\langle\lambda, \operatorname{div}\mathbf{u}'\rangle = -\langle\operatorname{grad}\lambda, \mathbf{u}'\rangle. \tag{7}$$

So $0 = \langle\mathbf{u}', \mathbf{u}-\mathbf{v}-\operatorname{grad}\lambda\rangle$. Assume that there is still enough freedom in the choice of $\mathbf{u}'$ to conclude that

$$\mathbf{u} = \mathbf{v} + \operatorname{grad}\lambda. \tag{8}$$

Substituting this into the constraint equation $\operatorname{div}\mathbf{u} = \mathbf{s}$ gives

$$\operatorname{div}\operatorname{grad}\lambda = \mathbf{s} - \operatorname{div}\mathbf{v} \tag{9}$$

This is a necessary condition to have a minimum. To pick out a single solution, we need to impose boundary conditions on this abstract Poisson equation. We derive these boundary conditions by attempting to show that $\mathbf{u}$ satisfying (8) and (9) is the minimum and seeing what additional assumptions we need.

Let $\mathbf{f}$ be another vector that satisfies the discrete divergence condition $\operatorname{div}\mathbf{f} = \mathbf{s}$, and write $\mathbf{f} = \mathbf{u}+\mathbf{w}$. So $\operatorname{div}\mathbf{w} = 0$. $\mathbf{f} = \mathbf{v} + (\operatorname{grad}\lambda) + \mathbf{w}$. Then $\|\mathbf{f}-\mathbf{v}\|^2 = \|\operatorname{grad}\lambda + \mathbf{w}\|^2 = \langle\operatorname{grad}\lambda+\mathbf{w}, \operatorname{grad}\lambda+\mathbf{w}\rangle = \|\operatorname{grad}\lambda\|^2 + 2\langle\operatorname{grad}\lambda,\mathbf{w}\rangle + \|\mathbf{w}\|^2$. Since we could replace $\mathbf{w}$ with its opposite, this is minimized at $\mathbf{w} = 0$ if and only if

$$\langle\operatorname{grad}\lambda, \mathbf{w}\rangle = 0. \tag{10}$$

So the key to minimizing distance is ***orthogonal*** **projection**. Indeed, observe that

$$\langle\mathbf{u}-\mathbf{v}, \mathbf{w}\rangle = \langle\operatorname{grad}\lambda, \mathbf{w}\rangle = 0 \tag{11}$$

C.3 Statement of continuum problem

We recall the continuum problem:
Let $\mathbf{v}$ be a vector field on the domain Ω.
Let σ be a desired divergence.
Find $\mathbf{u}$ that minimizes $\|\mathbf{u}-\mathbf{v}\|^2$ subject to the constraint that $\nabla\cdot\mathbf{u} = \sigma$ in Ω.

C.4 Solution

Use Lagrange multipliers.
Let $L(\mathbf{u},\lambda) = \int_\Omega(\mathbf{u}-\mathbf{v})^2 + \int_\Omega 2\lambda(\nabla\cdot\mathbf{u}-\sigma)$, where λ is a Lagrange multiplier function, and where $\operatorname{dom}(\lambda) = \Omega$.
We minimize L.

Perturbing the multipliers simply recovers the constraints.
Let $\mathbf{u}'$ be a test perturbation.
$d_\epsilon|_{\epsilon=0} L(\mathbf{u}+\epsilon\mathbf{u}') = \int_\Omega 2(\mathbf{u}-\mathbf{v})\cdot\mathbf{u}' + \int_\Omega 2\lambda\nabla\cdot\mathbf{u}'$.
The left hand side should be zero. If we impose $\mathbf{u}' = 0$ on $\partial\Omega$, we get: $0 = \int_\Omega 2(\mathbf{u}-\mathbf{v})\cdot\mathbf{u}' - \int_\Omega 2\nabla\lambda\cdot\mathbf{u}'$.

Since $\mathbf{u}'$ is arbitrary, we get $0 = 2(\mathbf{u}-\mathbf{v}) - 2\nabla\lambda$, i.e.,

$$\mathbf{u} = \mathbf{v} + \nabla\lambda \text{ in } \Omega. \tag{12}$$

Substituting this into the constraint equations gives the equation:

$$\nabla^2\lambda = \sigma - \nabla\cdot\mathbf{v} \text{ in } \Omega. \tag{13}$$

This is a necessary condition to have a minimum. To pick out a single solution, we need to impose boundary conditions on this Poisson equation. We derive these boundary conditions by attempting to show that $\mathbf{u}$ satisfying equations (12) and (13) is the minimum and seeing what additional assumptions are necessary.

Let $\mathbf{f}$ be another function that satisfies $\nabla\cdot\mathbf{f} = \sigma$, and write $\mathbf{f} = \mathbf{u}+\mathbf{w} = \mathbf{v}+\nabla\lambda+\mathbf{w}$. So $\nabla\cdot\mathbf{w} = 0$. Then $\|\mathbf{f}-\mathbf{v}\|^2 = \|\nabla\lambda+\mathbf{w}\|^2 = \langle\nabla\lambda+\mathbf{w}, \nabla\lambda+\mathbf{w}\rangle = \|\nabla\lambda\|^2 + 2\langle\nabla\lambda,\mathbf{w}\rangle + \|\mathbf{w}\|^2$. Since we could replace $\mathbf{w}$ with its opposite, this is minimized at $\mathbf{w} = 0$ if and only if $\langle\nabla\lambda,\mathbf{w}\rangle = 0$. So the key to minimizing distance is ***orthogonal*** **projection**.

But $\langle\nabla\lambda,\mathbf{w}\rangle = \int_\Omega \nabla\lambda\cdot\mathbf{w} = \int_\Omega \nabla\cdot(\lambda\mathbf{w}) - \int_\Omega \lambda\underbrace{\nabla\cdot\mathbf{w}}_{0} = \oint_{\partial\Omega}\mathbf{n}\cdot(\lambda\mathbf{w})$, which equals 0 if we impose homogeneous Dirichlet boundary conditions, i.e., $\lambda = 0$ on $\partial\Omega$.