Mathematical entropy, vanishing viscosity, and symmetrizability

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This note explains the relationship between symmetric hyperbolic form and the existence of an entropy. It

1. defines mathematical entropy,
2. shows that a viscosity solution satisfies the entropy inequality,
3. shows that symmetrizability of conservation law is equivalent to the existence of a mathematical entropy, and
4. shows that symmetrizability of a conservation law implies hyperbolicity (well-posedness).

Below x^i are (spatial) coordinates, and the repeated index i indicates summation from 1 to 3. Subscripts are used to denote partial derivatives.

We study the conservation law

$$u_t + f^i(u)_{x^i} = 0.$$

1 Entropy, symmetric variables, and hyperbolicity

To seek an entropy ϕ, we multiply and dot the quasilinear form with ϕ_u:

$$\phi_u \cdot u_t + \phi_u \cdot f^i_u \cdot u_{x^i} = 0.$$

Suppose that we can find $\phi(u)$ and $F^i(u)$ such that

$$F^i_u = \phi_u \cdot f^i_u.$$

Then entropy is conserved:

$$\phi_t + F^i_{x^i} = 0.$$

Observation: in an adiabatic system the value of the "entropy" $s = \log(p\rho^{-\gamma})$ is conserved along particle paths:

$$d_t(s) = 0.$$

This means that $\phi := (\rho s)$ is entropy in the sense above of having an entropy flux:

$$(\rho s)_t + \nabla \cdot (\mathbf{u}\rho s) = 0.$$

1.1 A convex entropy is satisfied by vanishing viscosity

If we also assume that ϕ_{uu} is positive definite, we can show that

$$\phi_t + F^i_{x^i} \leq 0 \text{ (in the sense of distributions).}$$

The inequality is an equality for smooth solutions. (I understand that this requirement uniquely picks out the vanishing viscosity solution from other weak solutions – for a proof in the case of scalar conservation laws see section 11.4 (Entropy Criteria) of *Partial Differential Equations*, by Lawrence C. Evans (1998).)

To see this, we consider the viscosity solution

$$u_t + f^i_{x^i} = \epsilon u_{x^i x^i}$$

and assume that, as ϵ goes to zero, u is uniformly bounded and converges almost everywhere.

As before, we convert the viscosity solution to a (viscous) entropy evolution equation by taking the dot product with ϕ_u:

$$\phi_t + f^i_{x^i} = \epsilon \phi_u \cdot u_{x^i x^i}.$$

Looking for a "chain rule" to simplify the RHS (or looking for a product rule to move derivatives from u) prompts the observation

$$\phi_{x^i} = \phi_u \cdot u_{x^i} \text{ and}$$

$$\phi_{x^i x^i} = (\phi_{uu} \cdot u_{x^i}) \cdot u_{x^i} + \phi_u \cdot u_{x^i x^i},$$

so the entropy evolution equation becomes

$$\phi_t + f^i_{x^i} = \epsilon(\phi_{x^i x^i} - (\phi_{uu} \cdot u_{x^i}) \cdot u_{x^i}).$$

Invoking that ϕ_{uu} is positive definite,

$$(\phi_{uu} \cdot u_{x^i}) \cdot u_{x^i} \geq 0,$$

so we get the viscous entropy evolution inequality

$$\phi_t + f^i_{x^i} - \epsilon \phi_{x^i x^i} \leq 0.$$

Roughly, taking $\epsilon \to 0$ gives $\phi_t + f^i_{x^i} \leq 0$. More rigorously, to show that this holds in the sense of distributions, we multiply the viscous entropy inequality by a test function v and integrate (by parts) over time $T = [0, \infty)$ and the spatial domain X:

$$\int_T \int_X (\phi v_t + f^i v_{x^i} + \epsilon \phi v_{x^i x^i}) \geq 0.$$

Assuming that u is uniformly bounded (almost everywhere by a measurable function independent of ϵ) and converges almost everywhere as ϵ goes to zero, the dominated convergence theorem allows us to bring the limit inside the integral and gives

$$\int_T \int_X (\phi v_t + f^i v_{x^i}) \geq 0,$$

which is what is meant by the statement

$$\phi_t + f^i_{x^i} \leq 0 \text{ (in the sense of distributions).}$$

1.2 Symmetrizability ensures existence of entropy

Given the conservation law

$$u_t + f^i_{x^i} = 0,$$

we seek symmetric variables v. The chain rule says that

$$u_v \cdot v_t + f^i_v \cdot v_{x^i} = 0.$$

Suppose that u_v and f^i_v are symmetric. Then (by modifying rectilinear paths in only two variables at a time and invoking Green's theorem) path integrals of u_v and f^i_v are independent of (rectilinear) path and can be used to define scalar potentials ϕ^* and r^i satisfying

$$u = \phi^*_v \text{ and}$$
$$f^i = r^i_v,$$

where we note that ϕ^* is convex if u_v is positive definite; in this case $u(v)$ is injective and we can speak of $v(u)$.

The Legendre transform (which is convexity-preserving),

$$\phi(u) = u \cdot v - \phi^*(v),$$

and the "generalized Legendre transform"

$$F^i(u) = f^i \cdot v - r^i(v)$$

then allow us to write

$$v = \phi_u,$$

where ϕ_{uu} is positive definite, and

$$F^i_u = v \cdot f^i_u = \phi_u \cdot f^i_u,$$

which satisfies the conditions in the entropy framework to conclude that

$$\phi_t + F^i_{x^i} \leq 0,$$

as needed.

We remark that the entropy is the Legendre transform of the potential of the state with respect to the symmetric variables. (In other words, the derivative of the entropy is the inverse of the derivative of the potential function of the state variables.) Similarly, the entropy fluxes are the "generalized Legendre transform" of the potential of the fluxes with respect to the symmetric variables.

1.3 Entropy ensures existence of symmetric variables.

Suppose that the conservation law

$$u_t + f^i_{x^i} = 0$$

satisfies an entropy inequality

$$\phi_t + F^i_{x^i} \leq 0,$$

where $\phi(u)$ and $F^i(u)$ are scalars, the inequality is an equality for smooth solutions, and ϕ is a convex function of u (e.g. $\phi_{uu} > 0$). We will show that the variables $v := \phi_u$ are symmetric variables.

Assuming smoothness, the entropy equality says

$$\phi_u \cdot u_t + F^i_u \cdot u_{x^i} = 0,$$

and the conservation law, multiplied by ϕ_u, says

$$\phi_u \cdot u_t + \phi_u \cdot f^i_u \cdot u_{x^i} = 0.$$

Matching these last two equations reveals that any entropy flux must satisfy

$$F^i_u = \phi_u \cdot f^i_u.$$

We define the symmetric variables by

$$v := \phi_u. \quad (\text{So } F^i_u = v \cdot f^i_u.)$$

The assumption that ϕ is convex says that $v(u)$ is injective and we can speak of $u(v)$. Using a Legendre transformation,

$$u = \phi^*_v, \text{ where } \phi^*(v) := v \cdot u - \phi.$$

Applying the chain rule to the conservation law,

$$u_v \cdot v_t + f^i_v \cdot v_{x^i} = 0.$$

Since v_u is symmetric positive definite, so is u_v. I claim that f^i_v is symmetric, i.e., that f^i is the gradient of a potential,

$$f^i = r^i_v.$$

Indeed, using the generalized Legendre transform we make the definition

$$r^i(v) := v \cdot f^i - F^i(u)$$

and verify that

$$r^i_v = f^i + v \cdot f^i_u \cdot u_v - F^i_u \cdot u_v = f^i,$$

as needed.

1.4 Symmetric variables ensure hyperbolicity.

Suppose that for the conservation law

$$u_t + f^i_{x^i} = 0$$

symmetric variables v exist such that u_v is symmetric positive definite and f^i_v is symmetric.

I claim that the conservation law is hyperbolic. This means that (assuming smoothness) in the quasilinearization

$$u_t + f^i_u \cdot u_{x^i} = 0$$

each matrix f^i_u has real eigenvalues and a full set of eigenvectors.

Indeed, assume smoothness and express the conservation law in terms of symmetric variables,

$$u_v \cdot v_t + f^i_v \cdot v_{x^i} = 0,$$

where u_v is symmetric positive definite and f^i_v is symmetric. So

$$\sqrt{u_v} \cdot v_t + (\sqrt{v_u} \cdot f^i_v \cdot \sqrt{v_u}) \cdot (\sqrt{u_v} \cdot v_{x^i}) = 0,$$

i.e.,

$$A \cdot v_t + C^i \cdot A \cdot v_{x^i} = 0,$$

where $A := \sqrt{u_v}$ and where $C^i := \sqrt{v_u} \cdot f^i_v \cdot \sqrt{v_u}$.

We now linearize by freezing the coefficients about a state v_0 and replacing v by $(dv) := v - v_0$:

$$A \cdot (dv)_t + C^i \cdot A \cdot (dv)_{x^i} = 0.$$

Defining the variables

$$w := A \cdot (dv),$$

this says

$$w_t + C^i \cdot w_{x^i} = 0,$$

where (recall) $C^i := \sqrt{v_u} \cdot f^i_v \cdot \sqrt{v_u}$. Since $\sqrt{v_u}$ and f^i_v are symmetric, so is the (frozen) coefficient matrix C^i, so this linearized system has real eigenvalues and a full set of (orthogonal) eigenvectors.

I claim that this implies that f^i_u has real eigenvalues and a full set of eigenvectors. This follows because (1) the evolution equation for u has a linearization

$$(du)_t + f^i_u \cdot (du)_{x^i}$$

whose frozen coefficients are the matrix f^i_u, and (2) the variables (du) are related to the variables w by an invertible linear transformation: since $w = A \cdot (dv)$ and $(dv) = v_u \cdot (du)$, $w = B \cdot (du)$ where $B := (A \cdot v_u)$.

Plugging this into the evolution equation for w,

$$(du)_t + B^{-1} \cdot C^i \cdot B \cdot (du)_{x^i} = 0;$$

matching up these linearizations for the evolution of (du) shows that

$$f^i_u = B^{-1} \cdot C^i \cdot B.$$

So f^i_u is similar to C^i and therefore has the same eigenvalues and has a full set of eigenvectors.

1.5 Entropy ensures hyperbolicity.

The existence of an entropy ϕ for a conservation law ensures that symmetric variables $v := \phi_u$ exist. But the existence of symmetric variables ensures hyperbolicity.

1.6 Hyperbolicity does not ensure symmetrizability?

(Need to cook up an example.)