# Translation between Heaviside-Lorentz (HL), Gaussian (cgs), and SI (mks or "rationalized") systems of units

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## 1 Definitions

| variable | meaning |
|---|---|
| $n$ | particle density |
| $q$ | charge per particle |
| $m$ | mass per particle |
| $\mathbf{v}(\mathbf{x}, t)$ | velocity of particles |
| $d_t = \partial_t + \mathbf{v} \cdot \nabla$ | convective derivative |

| variable | meaning |
|---|---|
| $\mathbf{B}$ | magnetic field |
| $\mathbf{E}$ | electric field |
| $c$ | speed of light |
| $\epsilon_0$ | permittivity of free space |

One way to determine how to relate quantities in different units is to make the equations of the one system look like the equations of the corresponding system. In this note we take this approach to relate Heaviside-Lorentz, Gaussian, SI units. For simplicity we derive dimension transformations using Maxwell's equations with the momentum equation for a cold one-species plasma, rather than with the momentum equation for individual particles or with the Boltzmann equation.

## 2 Heaviside-Lorentz (HL) $\longleftrightarrow$ Gaussian

| **Heaviside-Lorentz** | **Gaussian-looking** |
|---|---|
| $\nabla \cdot \mathbf{E} = qn$ | $\nabla \cdot (\sqrt{4\pi}\mathbf{E}) = 4\pi\left(\frac{q}{\sqrt{4\pi}}\right)n$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot (\sqrt{4\pi}\mathbf{B}) = 0$ |
| $\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$ | $\partial_t(\sqrt{4\pi}\mathbf{B}) + c\nabla \times (\sqrt{4\pi}\mathbf{E}) = 0$ |
| $\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -qn\mathbf{v}$ | $\partial_t(\sqrt{4\pi}\mathbf{E}) - c\nabla \times (\sqrt{4\pi}\mathbf{B}) = -4\pi\left(\frac{q}{\sqrt{4\pi}}\right)n\mathbf{v}$ |
| $mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $mnd_t\mathbf{v} = \left(\frac{q}{\sqrt{4\pi}}\right)n((\sqrt{4\pi}\mathbf{E}) + \frac{\mathbf{v}}{c} \times (\sqrt{4\pi}\mathbf{B}))$ |

| **Gaussian** | **Heaviside-Lorentz-looking** |
|---|---|
| $\nabla \cdot \mathbf{E} = 4\pi qn$ | $\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) = (\sqrt{4\pi}q)n$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right) = 0$ |
| $\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$ | $\partial_t\left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right) + c\nabla \times \left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) = 0$ |
| $\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -4\pi qn\mathbf{v}$ | $\partial_t\left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) - c\nabla \times \left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right) = -(\sqrt{4\pi}q)n\mathbf{v}$ |
| $mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $mnd_t\mathbf{v} = (\sqrt{4\pi}q)n(\left(\frac{\mathbf{E}}{\sqrt{4\pi}}\right) + \frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{B}}{\sqrt{4\pi}}\right))$ |

$$\mathbf{E}_{\text{gauss}} = \sqrt{4\pi}\mathbf{E}_{\text{HL}}, \qquad \mathbf{E}_{\text{HL}} = \frac{\mathbf{E}_{\text{gauss}}}{\sqrt{4\pi}},$$

$$\mathbf{B}_{\text{gauss}} = \sqrt{4\pi}\mathbf{B}_{\text{HL}}, \qquad \mathbf{B}_{\text{HL}} = \frac{\mathbf{B}_{\text{gauss}}}{\sqrt{4\pi}},$$

$$q_{\text{gauss}} = \frac{q_{\text{HL}}}{\sqrt{4\pi}}, \qquad q_{\text{HL}} = \sqrt{4\pi}q_{\text{gauss}}.$$

## 3 Heaviside-Lorentz (HL) $\longleftrightarrow$ SI

| **Heaviside-Lorentz** | **SI-looking** |
|---|---|
| $\nabla \cdot \mathbf{E} = qn$ | $\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) = \frac{\left(\sqrt{\epsilon_0} q\right) n}{\epsilon_0}$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right) = 0$ |
| $\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$ | $\partial_t \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right) + \nabla \times \left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) = 0$ |
| $\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -qn\mathbf{v}$ | $\partial_t \left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) - c^2 \nabla \times \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right) = -\frac{\left(\sqrt{\epsilon_0} q\right) n\mathbf{v}}{\epsilon_0}$ |
| $mnd_t \mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $mnd_t \mathbf{v} = \left(\sqrt{\epsilon_0} q\right) n\left(\left(\frac{\mathbf{E}}{\sqrt{\epsilon_0}}\right) + \mathbf{v} \times \left(\frac{\mathbf{B}}{c\sqrt{\epsilon_0}}\right)\right)$ |

| **SI** | **Heaviside-Lorentz-looking** |
|---|---|
| $\nabla \cdot \mathbf{E} = \frac{qn}{\epsilon_0}$ | $\nabla \cdot (\sqrt{\epsilon_0} \mathbf{E}) = \left(\frac{q}{\sqrt{\epsilon_0}}\right) n$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot (c\sqrt{\epsilon_0} \mathbf{B}) = 0$ |
| $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$ | $\partial_t (c\sqrt{\epsilon_0} \mathbf{B}) + c\nabla \times (\sqrt{\epsilon_0} \mathbf{E}) = 0$ |
| $\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\frac{qn\mathbf{v}}{\epsilon_0}$ | $\partial_t (\sqrt{\epsilon_0} \mathbf{E}) - c\nabla \times (c\sqrt{\epsilon_0} \mathbf{B}) = -\left(\frac{q}{\sqrt{\epsilon_0}}\right) n\mathbf{v}$ |
| $mnd_t \mathbf{v} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ | $mnd_t \mathbf{v} = \left(\frac{q}{\sqrt{\epsilon_0}}\right) n((\sqrt{\epsilon_0} \mathbf{E}) + \frac{\mathbf{v}}{c} \times (c\sqrt{\epsilon_0} \mathbf{B}))$ |

$$\mathbf{E}_{\mathrm{HL}} = \sqrt{\epsilon_0} \mathbf{E}_{\mathrm{SI}}, \qquad \mathbf{E}_{\mathrm{SI}} = \frac{\mathbf{E}_{\mathrm{HL}}}{\sqrt{\epsilon_0}},$$

$$\mathbf{B}_{\mathrm{HL}} = c\sqrt{\epsilon_0} \mathbf{B}_{\mathrm{SI}}, \qquad \mathbf{B}_{\mathrm{SI}} = \frac{\mathbf{B}_{\mathrm{HL}}}{c\sqrt{\epsilon_0}},$$

$$q_{\mathrm{HL}} = \frac{q_{\mathrm{SI}}}{\sqrt{\epsilon_0}}. \qquad q_{\mathrm{SI}} = \sqrt{\epsilon_0} q_{\mathrm{HL}}.$$

Notice that there is an SI-looking system for every choice of $\epsilon_0$. Choosing $\epsilon_0 = 1$ makes it simple to convert from SI to HL formulas: just replace $\mathbf{B}$ with $\mathbf{B}/c$ and drop $\epsilon_0$.

## 4 SI $\longleftrightarrow$ Gaussian

| SI | Gaussian-looking |
|---|---|
| $\nabla \cdot \mathbf{E} = \frac{qn}{\epsilon_0}$ | $\nabla \cdot (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 4\pi\left(\frac{q}{\sqrt{4\pi\epsilon_0}}\right)n$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = 0$ |
| $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$ | $\partial_t (c\sqrt{4\pi\epsilon_0}\mathbf{B}) + c\nabla \times (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 0$ |
| $\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\frac{qn\mathbf{v}}{\epsilon_0}$ | $\partial_t(\sqrt{4\pi\epsilon_0}\mathbf{E}) - c\nabla \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = -4\pi\left(\frac{q}{\sqrt{4\pi\epsilon_0}}\right)n\mathbf{v}$ |
| $mnd_t\mathbf{v} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ | $mnd_t\mathbf{v} = \left(\frac{q}{\sqrt{4\pi\epsilon_0}}\right)n((\sqrt{4\pi\epsilon_0}\mathbf{E}) + \frac{\mathbf{v}}{c} \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}))$ |

| Gaussian | SI-looking |
|---|---|
| $\nabla \cdot \mathbf{E} = 4\pi qn$ | $\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = \frac{(\sqrt{4\pi\epsilon_0}q)n}{\epsilon_0}$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = 0$ |
| $\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$ | $\partial_t\left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) + \nabla \times \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = 0$ |
| $\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -4\pi qn\mathbf{v}$ | $\partial_t\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) - c^2\nabla \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = -\frac{(\sqrt{4\pi\epsilon_0}q)n\mathbf{v}}{\epsilon_0}$ |
| $mnd_t\mathbf{v} = qn(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $mnd_t\mathbf{v} = (\sqrt{4\pi\epsilon_0}q)n(\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) + \mathbf{v} \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right))$ |

$$\mathbf{E}_{\mathrm{SI}} = \frac{\mathbf{E}_{\mathrm{gauss}}}{\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{E}_{\mathrm{gauss}} = \sqrt{4\pi\epsilon_0}\mathbf{E}_{\mathrm{SI}},$$

$$\mathbf{B}_{\mathrm{SI}} = \frac{\mathbf{B}_{\mathrm{gauss}}}{c\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{B}_{\mathrm{gauss}} = c\sqrt{4\pi\epsilon_0}\mathbf{B}_{\mathrm{SI}},$$

$$q_{\mathrm{SI}} = \sqrt{4\pi\epsilon_0}q_{\mathrm{gauss}}. \qquad q_{\mathrm{gauss}} = \frac{q_{\mathrm{SI}}}{\sqrt{4\pi\epsilon_0}}.$$

Notice that there is an SI-looking system for every choice of $\epsilon_0$. Choosing $\epsilon_0$ so that $4\pi\epsilon_0 = 1$ makes it simple to convert from SI to Gaussian formulas: just replace $\mathbf{B}$ with $\mathbf{B}/c$ and replace $\epsilon_0$ with $\frac{1}{4\pi}$.

## 5 Plasma

We now consider what happens to the plasma equations under transformation between SI and Gaussian units.

# 6 Boltzmann

| SI | Gaussian-looking |
|---|---|
| $\nabla \cdot \mathbf{E} = \frac{\sigma}{\epsilon_0}$ | $\nabla \cdot (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 4\pi\left(\frac{\sigma}{\sqrt{4\pi\epsilon_0}}\right)$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = 0$ |
| $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$ | $\partial_t(c\sqrt{4\pi\epsilon_0}\mathbf{B}) + c\nabla \times (\sqrt{4\pi\epsilon_0}\mathbf{E}) = 0$ |
| $\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\frac{\mathbf{J}}{\epsilon_0}$ | $\partial_t(\sqrt{4\pi\epsilon_0}\mathbf{E}) - c\nabla \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}) = -4\pi\left(\frac{\mathbf{J}}{\sqrt{4\pi\epsilon_0}}\right)$ |
| $m d_t \mathbf{v}_p = q_p(\mathbf{E} + \mathbf{v}_p \times \mathbf{B})$ | $m d_t \mathbf{v}_p = \left(\frac{q_p}{\sqrt{4\pi\epsilon_0}}\right)((\sqrt{4\pi\epsilon_0}\mathbf{E}) + \frac{\mathbf{v}_p}{c} \times (c\sqrt{4\pi\epsilon_0}\mathbf{B}))$ |

| Gaussian | SI-looking |
|---|---|
| $\nabla \cdot \mathbf{E} = 4\pi\sigma$ | $\nabla \cdot \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = \frac{(\sqrt{4\pi\epsilon_0}\sigma)}{\epsilon_0}$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = 0$ |
| $\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = 0$ | $\partial_t\left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) + \nabla \times \left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) = 0$ |
| $\partial_t \mathbf{E} - c\nabla \times \mathbf{B} = -4\pi\mathbf{J}$ | $\partial_t\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) - c^2\nabla \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right) = -\frac{(\sqrt{4\pi\epsilon_0}\mathbf{J})}{\epsilon_0}$ |
| $m_p d_t \mathbf{v}_p = q_p(\mathbf{E} + \frac{\mathbf{v}_p}{c} \times \mathbf{B})$ | $m_p d_t \mathbf{v}_p = (\sqrt{4\pi\epsilon_0}q_p)(\left(\frac{\mathbf{E}}{\sqrt{4\pi\epsilon_0}}\right) + \mathbf{v}_p \times \left(\frac{\mathbf{B}}{c\sqrt{4\pi\epsilon_0}}\right))$ |

$$\mathbf{E}_{\text{SI}} = \frac{\mathbf{E}_{\text{gauss}}}{\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{E}_{\text{gauss}} = \sqrt{4\pi\epsilon_0}\mathbf{E}_{\text{SI}},$$

$$\mathbf{B}_{\text{SI}} = \frac{\mathbf{B}_{\text{gauss}}}{c\sqrt{4\pi\epsilon_0}}, \qquad \mathbf{B}_{\text{gauss}} = c\sqrt{4\pi\epsilon_0}\mathbf{B}_{\text{SI}},$$

$$\sigma_{\text{SI}} = \sqrt{4\pi\epsilon_0}\sigma_{\text{gauss}}. \qquad \sigma_{\text{gauss}} = \frac{\sigma_{\text{SI}}}{\sqrt{4\pi\epsilon_0}}.$$

$$\mathbf{J}_{\text{SI}} = \sqrt{4\pi\epsilon_0}\mathbf{J}_{\text{gauss}}. \qquad \mathbf{J}_{\text{gauss}} = \frac{\mathbf{J}_{\text{SI}}}{\sqrt{4\pi\epsilon_0}}.$$

## 7 Conclusion

I prefer to work with SI units because: (1) it is easier to convert from SI units to the other systems than to convert from either of the other systems, and (2) a generic nondimensionalization of the kinetic-Maxwell system yields a system that has the same form as the SI system (if time is nondimensionalized by the gyroperiod — see section A.2 of my PhD thesis).