Fundamental Equations of Plasmas,

by E. Alec Johnson, March 28, 2007

1 Full one-fluid plasma equations.

1.1 Definitions of Quantities.

n = particle number density

1.1.1 Electromagnetic quantities.

$\mathbf{B}$ = magnetic field
$\mathbf{E}$ = electric field
$\mathbf{S}$ = Poynting vector
ϵ_0 = permittivity of vacuum
μ_0 = permeability of vacuum
c = speed of light
σ = charge density
$\mathbf{J}$ = current density (net charge flux)
η = resistivity

1.1.2 Mechanical quantities.

ρ = net mass density
$\mathbf{v}$ = fluid velocity
$(\rho\mathbf{v})$ = momentum density (i.e. mass flux)
p = gas-dynamic pressure
$\underline{\underline{\sigma}}$ = viscous stress tensor
$\underline{\underline{\tau}}$ = total mechanical stress tensor
$\underline{\underline{T}}$ = stress of electromagnetic field
$\underline{\underline{e}}$ = deformation rate (strain rate, even part)
μ = shear viscosity
λ = "balancing bulk viscosity"

1.1.3 Thermodynamic quantities.

$\mathbf{q}$ = heat flux
T = temperature
κ = heat conductivity
γ = ratio of specific heats
$\mathcal{R}$ = gas constant
$\tilde{\mathcal{E}}$ = total energy density
$\mathcal{E}$ = gas-dynamic energy
$\mathcal{E}^t$ = thermal energy
$\mathcal{E}^k$ = kinetic energy
$\mathcal{E}^f$ = electromagnetic field energy

1.2 Defining and Constituting Relationships.

1.2.1 Electromagnetic relations.

$c^2\mu_0\epsilon_0 \equiv 1$
$\mathbf{S} := \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$
$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J}$

1.2.2 Mechanical relations.

$\underline{\underline{e}} := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$
$\underline{\underline{\sigma}} = \lambda\nabla\cdot\mathbf{v}\underline{\underline{\delta}} + 2\mu\underline{\underline{e}}$
$\lambda = -\frac{2}{3}\mu$ (assuming trace($\underline{\underline{\sigma}}$) = 0)
$\underline{\underline{\tau}} = -p\underline{\underline{\delta}} + \underline{\underline{\sigma}}$
$\underline{\underline{T}} := \epsilon_0(\mathbf{EE} - \frac{1}{2}E^2\underline{\underline{\delta}}) + \frac{1}{\mu_0}(\mathbf{BB} - \frac{1}{2}B^2\underline{\underline{\delta}})$

1.2.3 Thermodynamic relations.

$p = \rho\mathcal{R}T$
$\mathcal{E}^t = \frac{p}{\gamma-1}$
$\mathcal{E}^k := \frac{1}{2}\rho v^2$
$\mathcal{E}^f := \epsilon_0\frac{1}{2}E^2 + \frac{1}{2\mu_0}B^2$
$\mathbf{q} = -\kappa\nabla T$
$\tilde{\mathcal{E}} := \mathcal{E} + \mathcal{E}^f$.
$\mathcal{E} := \mathcal{E}^t + \mathcal{E}^k$.

1.3 Definitions of Symbols and Operators.

- $\underline{\underline{\underline{\epsilon}}}$ = permutation tensor
- $\partial_t := \frac{\partial}{\partial t}$
- $d_t := \partial_t + \mathbf{v}\cdot\nabla$ = **convective derivative**.
- $\delta_t := \alpha \mapsto (\partial_t\alpha + \nabla\cdot(\mathbf{v}\alpha))$ = **conservative derivative**.

1.4 Full One-fluid Plasma Balance Laws.

The full one-fluid plasma equations are a system of 11 equations which specify the evolution of electromagnetic field, mass density, momentum, and energy.

1.4.1 Electromagnetic evolution.

Maxwell's 6 evolution equations with the constraints that must be maintained by physical solutions are:

$$\partial_t\begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} + \nabla\times\begin{bmatrix}\mathbf{E}\\ -c^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}0\\ -\frac{1}{\epsilon_0}\mathbf{J}\end{bmatrix} \text{ and } \nabla\cdot\begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} = \begin{bmatrix}0\\ \frac{1}{\epsilon_0}\sigma\end{bmatrix}$$

1.4.2 Material balance laws (gas-dynamics)

The material balance laws are simply statements of conservation of mass, momentum, and energy. See appendix A for a derivation of the electromagnetic part.

$$\delta_t\begin{bmatrix}\rho\\ (\rho\mathbf{v})\\ \mathcal{E}\end{bmatrix} + \partial_t\begin{bmatrix}0\\ \frac{1}{c^2}\mathbf{S}\\ \mathcal{E}^f\end{bmatrix} + \nabla\cdot\begin{bmatrix}0\\ -\underline{\underline{\tau}}\\ -\underline{\underline{\tau}}\cdot\mathbf{v} + \mathbf{q}\end{bmatrix} + \nabla\cdot\begin{bmatrix}0\\ -\underline{\underline{T}}\\ \mathbf{S}\end{bmatrix} = 0$$

$$\boxed{\begin{aligned}&\text{i.e. } \delta_t\begin{bmatrix}\rho\\ (\rho\mathbf{v})\\ \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2\end{bmatrix} + \partial_t\begin{bmatrix}0\\ \epsilon_0\mathbf{E}\times\mathbf{B}\\ \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\end{bmatrix} + \nabla\cdot\begin{bmatrix}0\\ p\underline{\underline{\delta}}\\ p\mathbf{v}\end{bmatrix} = \\ &\nabla\cdot\begin{bmatrix}0\\ \underline{\underline{\sigma}}\\ \underline{\underline{\sigma}}\cdot\mathbf{v} + \kappa\nabla T\end{bmatrix} + \nabla\cdot\begin{bmatrix}0\\ \epsilon_0(\mathbf{EE} - \frac{E^2}{2}\underline{\underline{\delta}}) + \frac{1}{\mu_0}(\mathbf{BB} - \frac{B^2}{2}\underline{\underline{\delta}})\\ -\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\end{bmatrix}\end{aligned}}$$

2 Conservation laws for MHD.

The equations of MHD (Magnetohydrodynamics) are an approximation to the full one-fluid plasma equations above. The electric field $\mathbf{E}$ is eliminated by discarding $\partial_t\mathbf{E}$ (Ampere's magnetostatic approximation) and quadratic order electric field terms.

We will put each law in the form:

$$\boxed{\begin{aligned}&\partial_t(\text{conserved quantity}) + \nabla\cdot(\text{hyperbolic flux})\\ &= \nabla\cdot(\text{parabolic flux}).\end{aligned}}$$

2.1 Magnetic field.

The MHD equation for the evolution of $\mathbf{B}$ is obtained by using Ampere's law and Ohm's law in Faraday's law to eliminate $\mathbf{E}$ and $\mathbf{J}$:
$\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$
$\partial_t\mathbf{B} + \nabla\times(\mathbf{B}\times\mathbf{v} + \mathbf{E}') = 0$

$$\boxed{\partial_t\mathbf{B} + \nabla\cdot(\mathbf{vB} - \mathbf{Bv}) = -\nabla\times(\mathbf{E}') = \nabla\cdot(\underline{\underline{\underline{\epsilon}}}\cdot\mathbf{E}').}$$

Note: $-\nabla\times(\eta\mathbf{J}) = -\nabla\times(\eta\frac{1}{\mu_0}\nabla\times\mathbf{B})$
$= \nabla\cdot(\eta\frac{1}{\mu_0}(\nabla\mathbf{B}^T - \nabla\mathbf{B}))$.

2.2 Mass balance.

$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0.$

2.3 Momentum balance.

For MHD we ignore second-order terms in the electric field. This means that we discard the momentum of the electromagnetic field and retain only the magnetic terms in the electromagnetic stress tensor.
So the electromagnetic stress tensor is:
$\underline{\underline{T}} = \frac{1}{\mu_0}(\mathbf{BB} - \frac{1}{2}B^2\underline{\underline{\delta}})$

To see that we can discard the momentum of the electromagnetic field:
$\partial_t(\mathbf{E}\times\mathbf{B}) = (\partial_t\mathbf{E})\times\mathbf{B} + \mathbf{E}\times(\partial_t\mathbf{B})$
$= (\partial_t\mathbf{E})\times\mathbf{B} - \mathbf{E}\times\nabla\times\mathbf{E} \approxeq 0$

Decompose the stress tensor into its diagonal component (pressure) and its traceless component (viscous stress): $\underline{\underline{\tau}} = p\underline{\underline{\delta}} + \underline{\underline{\sigma}}$.

Now substitute into the general momentum balance
$\delta_t(\rho\mathbf{v}) + \partial_t(\epsilon_0\mathbf{E}\times\mathbf{B}) = \nabla\cdot\underline{\underline{\tau}} + \nabla\cdot\underline{\underline{T}}$.

Splitting the stress tensor into hyperbolic (pressure) and parabolic (viscous stress tensor) parts, we express conservation of momentum as:

$$\boxed{\partial_t\rho\mathbf{v} + \nabla\cdot\Big(\rho\mathbf{vv} + (p + \frac{1}{2\mu_0}B^2)\underline{\underline{\delta}} - \frac{1}{\mu_0}\mathbf{BB}\Big) = \nabla\cdot\underline{\underline{\sigma}}}$$

2.4 Energy balance.

Using Ohm's law and Ampere's law we can express the Poynting vector in terms of the magnetic field:
$\mathbf{E}\times\mathbf{B} = (\mathbf{E}' + \mathbf{B}\times\mathbf{v})\times\mathbf{B}$
$= \mathbf{E}'\times\mathbf{B} + (B^2\mathbf{v} - \mathbf{BB}\cdot\mathbf{v})$

Again we discard the electric field term from the electromagnetic energy since it is second-order:
$\mathcal{E}^f = \frac{1}{2\mu_0}B^2$.

Invoke the relations
$\underline{\underline{\tau}} = -p\underline{\underline{\delta}} + \underline{\underline{\sigma}}$ and
$\mathbf{q} = -\kappa\nabla T$.

Substituting into the general energy balance,
$\delta_t\mathcal{E} + \partial_t\mathcal{E}^f + \nabla\cdot(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}) = \nabla\cdot(\underline{\underline{\tau}}\cdot\mathbf{v}) - \nabla\cdot\mathbf{q}$,

$\delta_t\mathcal{E} + \frac{1}{2\mu_0}\partial_t B^2 + \nabla\cdot\frac{1}{\mu_0}(B^2\mathbf{v} - \mathbf{BB}\cdot\mathbf{v}) + \nabla\cdot\Big(\mathbf{E}'\times\mathbf{B}\Big)$
$= \nabla\cdot(\underline{\underline{\sigma}}\cdot\mathbf{v}) - \nabla\cdot(p\mathbf{v}) + \nabla\cdot(\kappa\nabla T)$

Thus the energy balance with electric field expurgated and hyperbolic and parabolic terms separated is:

$$\boxed{\begin{aligned}&\partial_t\tilde{\mathcal{E}} + \nabla\cdot\Big((\tilde{\mathcal{E}} + p + \frac{1}{2\mu_0}B^2)\mathbf{v} - \frac{1}{\mu_0}\mathbf{BB}\cdot\mathbf{v}\Big) = \\ &\nabla\cdot(\underline{\underline{\sigma}}\cdot\mathbf{v}) + \nabla\cdot\Big(\kappa\nabla T - \mathbf{E}'\times\mathbf{B}\Big)\end{aligned}}$$

where $\tilde{\mathcal{E}} = \mathcal{E}^t + \frac{1}{2}\rho v^2 + \frac{1}{2\mu_0}B^2$ is the total energy.
Assuming the ideal gas law, $\mathcal{E}^t = \frac{p}{\gamma-1}$. Note that:
$-\eta\mathbf{J}\times\mathbf{B} \approxeq -\eta\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = \eta\frac{1}{\mu_0}(\nabla(\frac{1}{2}B^2)) - \mathbf{B}\cdot\nabla\mathbf{B}$
$= \eta\frac{1}{\mu_0}\nabla\cdot(\frac{1}{2}B^2\underline{\underline{\delta}} - \mathbf{B}\cdot\mathbf{B})$.

2.5 Full MHD system.

Thus, the full system of viscous, resistive MHD equations for an ideal conducting gas is

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ \tilde{\mathcal{E}}\\ \mathbf{B}\end{bmatrix} + \nabla\cdot\underbrace{\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{vv} + \tilde{p}\underline{\underline{\delta}} - \frac{1}{\mu_0}\mathbf{BB}\\ \mathbf{v}(\tilde{\mathcal{E}} + \tilde{p}) - \frac{1}{\mu_0}\mathbf{BB}\cdot\mathbf{v}\\ \mathbf{vB} - \mathbf{Bv}\end{bmatrix}}_{\text{hyperbolic flux}}\\ &= \nabla\cdot\underbrace{\begin{bmatrix}0\\ \underline{\underline{\sigma}}\\ \underline{\underline{\sigma}}\cdot\mathbf{v} + \Big(\kappa\nabla T - \mathbf{E}'\times\mathbf{B}\Big)\\ \underline{\underline{\underline{\epsilon}}}\cdot\mathbf{E}'\end{bmatrix}}_{\text{parabolic flux}}\\ &\text{and} \qquad \nabla\cdot\mathbf{B} = 0\,,\end{aligned}}$$

where ρ is the mass density, $\mathbf{v}$ is the fluid velocity field, $\tilde{\mathcal{E}} := \mathcal{E} + \frac{1}{2\mu_0}B^2$ is the total energy (gas-dynamic energy plus magnetic energy), $\mathbf{B}$ is the magnetic field, and $\tilde{p} := p + \frac{1}{2\mu_0}B^2$ is the total pressure (gas-dynamic pressure plus magnetic pressure). The gas-dynamic pressure is $p = (\gamma-1)(\mathcal{E} - \frac{1}{2}\rho v^2)$, where γ is the ratio of specific heats.

3 Two-fluid plasma equations.

The two-fluid plasma equations consist of 16 evolution equations which specify balance laws for electromagnetic field and the mass, momentum, and energy of each species of the plasma. They model the plasma as a negatively charged fluid of electrons and a positively charged fluid of ions which occupy the same space and interact with the electromagnetic field. In the collisionless case, it is assumed that the two fluids pass through one another freely with no direct interaction, and therefore influence one another only by means of their mutual interaction with the electromagnetic field. In more general models the two fluids may exert a drag force on one another.

Our general two-fluid model consists simply of gas dynamics for each of the two fluids, coupled to one another by drag force and heat transfer and coupled to Maxwell's equations by means of source terms consisting of the Lorentz force, the charge density, and the current and displacement currents.

The 10 gas dynamics equations in generality are:

$$\partial_t \begin{bmatrix} \rho_s \\ \rho_s \mathbf{v}_s \\ \mathcal{E}_s \end{bmatrix} + \nabla \cdot \underbrace{\begin{bmatrix} \rho_s \mathbf{v}_s \\ \rho_s \mathbf{v}_s \mathbf{v}_s \\ \mathcal{E}_s \mathbf{v}_s \end{bmatrix}}_{\text{advection}} = \nabla \cdot \begin{bmatrix} 0 \\ \underline{\underline{\tau}}_s \\ \underline{\underline{\tau}}_s \cdot \mathbf{v}_s - \mathbf{q}_s \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ \mathbf{R}_s \\ \mathbf{R}_s \cdot \mathbf{v}_s + Q_s \end{bmatrix}}_{\text{interactive source}} + \underbrace{\begin{bmatrix} 0 \\ \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} \\ \mathbf{J}_s \cdot \mathbf{E} \end{bmatrix}}_{\text{electromagnetic source}},$$

where s is the species index (i for ion, e for electron), ρ denotes mass density, $\mathbf{v}$ is the fluid velocity, $\mathcal{E}$ is the gas-dynamic energy, $\underline{\underline{\tau}}$ is the stress, $\mathbf{q}$ is the heat flux, σ is the charge density, $\mathbf{J}$ is the current, $\mathbf{R}_s$ is the drag force on species s from the other species, and Q_s is the heat transfer to species s from the other species.

Maxwell's 6 evolution equations with constraints are:

$$\partial_t \begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix} + \nabla \times \begin{bmatrix} \mathbf{E} \\ -c^2 \mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\epsilon_0} \mathbf{J} \end{bmatrix} \text{ and } \nabla \cdot \begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\epsilon_0} \sigma \end{bmatrix}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\sigma = \sum_s \sigma_s = \sum_s \frac{q_s}{m_s} \rho_s$ is the charge density, and $\mathbf{J} = \sum_s \mathbf{J}_s = \sum_s \frac{q_s}{m_s} \rho_s \mathbf{v}_s$ is the current density.

We remark here that Maxwell's evolution equations can be viewed as a conservation law for $\mathbf{B}$ and a balance law for $\mathbf{E}$ (with current providing a source term), because a curl, like any spatial differential operator, can be viewed as a divergence: $\underline{\nabla} \times \underline{v} = \partial_j \mathbf{e}_i \epsilon_{ijk} v_k = -\underline{\nabla} \cdot (\underline{\underline{\epsilon}} \cdot \underline{v})$.

The 10 gas dynamics equations expressed with hyperbolic and parabolic flux terms and with interactive and electromagnetic source terms are:

$$\boxed{\begin{aligned} &\partial_t \begin{bmatrix} \rho_s \\ \rho_s \mathbf{v}_s \\ \mathcal{E}_s \end{bmatrix} + \nabla \cdot \underbrace{\begin{bmatrix} \rho_s \mathbf{v}_s \\ \rho_s \mathbf{v}_s \mathbf{v}_s + p_s \underline{\underline{\delta}} \\ \mathbf{v}_s (\mathcal{E}_s + p_s) \end{bmatrix}}_{\text{hyperbolic flux}} = \nabla \cdot \underbrace{\begin{bmatrix} 0 \\ \underline{\underline{\sigma}}_s \\ \underline{\underline{\sigma}}_s \cdot \mathbf{v}_s + \kappa_s \nabla T_s \end{bmatrix}}_{\text{parabolic flux}} \\ &+ \underbrace{\begin{bmatrix} 0 \\ \mathbf{R}_s \\ \mathbf{R}_s \cdot \mathbf{v}_s + Q_s \end{bmatrix}}_{\text{interactive source}} + \underbrace{\begin{bmatrix} 0 \\ \frac{q_s}{m_s} \rho_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \\ \frac{q_s}{m_s} \rho_s \mathbf{v}_s \cdot \mathbf{E} \end{bmatrix}}_{\text{electromagnetic source}} \end{aligned}}$$

where $\frac{q_s}{m_s}$ denotes charge-to-mass ratio, p is the pressure, $\underline{\underline{\sigma}}$ is the viscous stress, T is the temperature, and κ is the heat conductivity.

Typically $\mathbf{R}_s$ is taken to be proportional to the density of each species and the difference in velocity between the two species. Q_s is similarly proportional to the density of each species and the difference in temperature between them.

In the collisionless model, the interactive source is assumed to be zero. In the ideal model, the parabolic flux is also assumed to be zero. In the absence of shocks I think that we can then replace energy conservation with entropy conservation:
$d_t^s S_s = 0$, where $S_s := \ln(p_s \rho_s^{-\gamma})$

3.1 One-fluid from two-fluid.

To obtain the full one-fluid model from the two-fluid model, we simply sum the gas-dynamics balance laws over all species for each conserved variable.
(So let $\rho := \sum_s \rho_s$, $(\rho \mathbf{v}) := \sum_s (\rho \mathbf{v})_s$, $\mathcal{E} := \sum_s \mathcal{E}_s$, $\underline{\underline{\tau}} := \sum_s \underline{\underline{\tau}}_s$, $\mathbf{q} := \sum_s \mathbf{q}_s$, and $\mathbf{J} := \sum_s \mathbf{J}_s$.)

Interactive source terms will cancel, since they simply serve to exchange momentum and energy between species. The species index s will effectively disappear, except for quadratic deviations from the mean arising from the nonlinear term labeled "advection"; these nonlinearities can be absorbed into the higher-order moments (the stress tensor in the case of momentum conservation; the heat flux in the case of energy conservation). The full one-fluid model is only an approximation to the two-fluid model, because it assumes that nice constitutive relations for these higher-order moments still hold after absorbing these nonlinearities. [For details see my summary, "A book-keeping derivation of 1-fluid equations from multi-fluid plasma equations".]

A Derivation of basic laws.

A.1 Conservation of momentum.

The electromagnetic force on a particle of charge q and velocity $\mathbf{v}$ is given by: $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. This means that the electromagnetic force density on a continuum of net charge density σ and net current $\mathbf{J}$ is given by $\mathbf{F} = \sigma \mathbf{E} + \mathbf{J} \times \mathbf{B}$. (To see this, let n be the number density and $\mathbf{v}$ be the velocity of a particular species of charge q. Then the charge density of this species is $\sigma = nq$ and the current of this species is the charge flux, $\mathbf{J} = \sigma \mathbf{v} = nq\mathbf{v}$.)

Conservation of momentum tells us that:

$\delta_t(\rho \mathbf{v}) = \mathbf{F} + \nabla \cdot \underline{\underline{\tau}}$

We wish to write the force of the electromagnetic field on the particles as the time derivative of some function of electromagnetic field (which we will regard as electromagnetic momentum) plus a spatial derivative of another function of electromagnetic field (which we will regard as flux of electromagnetic momentum).

To express the force purely in terms of electromagnetic field quantities, use the nonhomogeneous Maxwell equations to eliminate the charge density and the current: $\mathbf{F} = \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + (\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \partial_t \mathbf{E}) \times \mathbf{B}$.

Then use parts to get a time derivative of a single quantity. $-(\partial_t \mathbf{E}) \times \mathbf{B} = -\partial_t (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \partial_t \mathbf{B}$.

The quantity $\epsilon_0 \mathbf{E} \times \mathbf{B} = \frac{1}{c^2} \mathbf{S}$, where $\mathbf{S}$ is the Poynting vector, is what we identify as the momentum of the field.

Now we'll use the inhomogeneous equations to make everything else look like the spatial derivative of a single quantity. Faraday's law gives $\mathbf{E} \times \partial_t \mathbf{B} = -\mathbf{E} \times (\nabla \times \mathbf{E})$. Now we try to write everything except the time derivative as the divergence of some tensor. For the electric field terms we get:

$$\begin{aligned} &(\nabla \cdot \mathbf{E}) \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E}) \\ &\quad = (\nabla \cdot \mathbf{E}) \mathbf{E} - (\nabla \mathbf{E}) \cdot \mathbf{E} + \mathbf{E} \cdot (\nabla \mathbf{E}) \\ &\quad = \nabla \cdot (\mathbf{E}\mathbf{E}) - \nabla(\tfrac{1}{2} E^2) \end{aligned}$$

For the magnetic field terms we get (since $\nabla \cdot \mathbf{B} = 0$):
$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{B} - \nabla(\frac{1}{2} B^2) = \nabla \cdot (\mathbf{B}\mathbf{B} - \frac{1}{2} B^2 \underline{\underline{\delta}})$.

So the force of the field on the charges is
$\mathbf{F} = -\partial_t (\frac{1}{c^2} \mathbf{S}) + \nabla \cdot \underline{\underline{T}}$,
where $\underline{\underline{T}} := \epsilon_0 (\mathbf{E}\mathbf{E} - \frac{1}{2} E^2 \underline{\underline{\delta}}) + \frac{1}{\mu_0} (\mathbf{B}\mathbf{B} - \frac{1}{2} B^2 \underline{\underline{\delta}})$
is the Maxwell stress tensor.

$$\boxed{\delta_t(\rho \mathbf{v}) + \partial_t (\tfrac{1}{c^2} \mathbf{S}) = \nabla \cdot \underline{\underline{T}} + \nabla \cdot \underline{\underline{\tau}}}$$

A.2 Conservation of energy.

The power (rate of work) of an electromagnetic field on a moving charged particle is (force) $\cdot$ (velocity) $= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\mathbf{v} \cdot \mathbf{E}$. This means that the power density on a net current $\mathbf{J}$ is given by $\mathbf{J} \cdot \mathbf{E}$. (To see this, let n be the number density and $\mathbf{v}$ be the velocity of a particular species of charge q. Then the current of this species is the charge flux, $\mathbf{J} = nq\mathbf{v}$.)

Conservation of energy tells us that:

$\delta_t \mathcal{E} = \mathbf{J} \cdot \mathbf{E} + \nabla \cdot (\underline{\underline{\tau}} \cdot \mathbf{v}) - \nabla \cdot \mathbf{q}$

We wish to write the work of the electromagnetic field on the particles as the time derivative of some function of electromagnetic field (which we will regard as electromagnetic momentum) plus a spatial derivative of another function of electromagnetic field (which we will regard as flux of electromagnetic energy).

To express the work purely in terms of electromagnetic field quantities, use the completed Ampere's law to eliminate the current, and then use parts and Faraday's law to separate out a time and spatial derivative:

$$\begin{aligned} -\mathbf{J} \cdot \mathbf{E} &= \epsilon_0 (\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B}) \cdot \mathbf{E} \\ &= \epsilon_0 (\partial_t (\tfrac{1}{2} E^2) - c^2 \mathbf{E} \cdot \nabla \times \mathbf{B}) \\ &= \epsilon_0 \partial_t (\tfrac{1}{2} E^2) - \tfrac{1}{\mu_0} (\mathbf{B} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{B})) \\ &= \epsilon_0 \partial_t (\tfrac{1}{2} E^2) + \tfrac{1}{\mu_0} \partial_t (\tfrac{1}{2} B^2) + \tfrac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) \\ &= \partial_t \Big(\underbrace{\epsilon_0 (\frac{1}{2} E^2) + \frac{1}{\mu_0} (\frac{1}{2} B^2)}_{\text{Call } \mathcal{E}^f} \Big) + \nabla \cdot \Big(\underbrace{\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}}_{\text{Call } \mathbf{S}} \Big) \end{aligned}$$

$$\boxed{\delta_t \mathcal{E} + \partial_t \mathcal{E}^f + \nabla \cdot \mathbf{S} = \nabla \cdot (\underline{\underline{\tau}} \cdot \mathbf{v}) - \nabla \cdot \mathbf{q}}$$

B Ohm's law.

Ohm's law specifies the electric field $\mathbf{E}' := \mathbf{E} + \mathbf{v} \times \mathbf{B}$ in the reference frame of the fluid. (In the approximation of Galilean relativity, the transformation of electromagnetic field from the fixed reference frame to a reference frame moving at velocity $\mathbf{v}$ is given by $\mathbf{B} \to \mathbf{B}$, $\mathbf{E} \to (\mathbf{E} + \mathbf{v} \times \mathbf{B})$.) Assuming quasineutrality and vanishing electron mass implies that the electron velocity in the reference frame of the ions is $\mathbf{w}_e := -\mathbf{J}/(en)$. Then conservation of momentum for electrons yields the generalized Ohm's law:

$$\boxed{\mathbf{E}' = \eta \mathbf{J} + \tfrac{\mathbf{J}}{en} \times \mathbf{B} - \tfrac{\nabla p_e}{en} + \tfrac{m_e}{e^2 n} [\partial_t \mathbf{J} + \nabla \cdot (\mathbf{J}\mathbf{v} + \mathbf{v}\mathbf{J} - \tfrac{\mathbf{J}\mathbf{J}}{ne})]}$$

Note that $\mathbf{E}' - \frac{\mathbf{J}}{en} \times \mathbf{B}$ is the electric field in the reference frame of the electrons. The final term represents the inertia.