General Moment Evolution

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1 Boltzmann equation

Recall the Boltzmann equation,

$$\partial_t f_s + \nabla \cdot (\mathbf{v} f_s) + \nabla_{\widetilde{\mathbf{v}}} \cdot (\mathbf{a} f_s) = C_s,$$

where $\nabla\cdot := \nabla_{\mathbf{x}}\cdot$, $\mathbf{x}$ is position, $\mathbf{v}$ is velocity, $\widetilde{\mathbf{v}} = \gamma\mathbf{v}$ is proper velocity, where $\gamma := (1 + (\mathbf{v}/c)^2)^{-1/2}$, and $\mathbf{a} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is the rate of change of proper velocity with respect to time. (Ignore the blue text and wide tildes if you do not care about relativity.) Drop the subscript s.

2 Evolution of "conserved" moments

Let $\chi(\mathbf{v})$ be a generic moment. Multiply by χ and integrate by parts. Get the generic velocity moment evolution equation

$$\partial_t(\rho\langle\chi\rangle) + \nabla \cdot (\rho\langle\mathbf{v}\chi\rangle) = \rho\langle\mathbf{a} \cdot \nabla_{\widetilde{\mathbf{v}}}\chi\rangle + \int_{\widetilde{\mathbf{v}}} \chi C,$$

where for any moment χ the statistical average $\langle\chi\rangle$ is defined as the average over velocity space weighted by the distribution f: $\langle\chi\rangle := \frac{\int_{\widetilde{\mathbf{v}}} \chi f}{\int_{\widetilde{\mathbf{v}}} f}$, i.e. $\rho\langle\chi\rangle := \int_{\widetilde{\mathbf{v}}} \chi f$.[1] In this document products are by default tensor products and powers are by default tensor powers. Now choose $\chi^{[n]} = \mathbf{v}^n := \prod_{i=1}^n \mathbf{v}$. Let Sym be the map which takes a tensor and returns its symmetric part (obtained by summing over all permutations of the tensor subscripts and dividing by n-factorial, where n is the order of the tensor). But $\mathbf{a} \cdot \nabla_{\widetilde{\mathbf{v}}}\mathbf{v}^n = \sum_j \mathbf{a}_j \partial_{v_j} \operatorname{Sym}(\mathbf{v}^n) = n \operatorname{Sym}(\mathbf{a}\mathbf{v}^{n-1}) = \frac{q}{m} n \operatorname{Sym}\big(\mathbf{v}^{n-1}\mathbf{E} + \mathbf{v}^n \times \mathbf{B}\big)$ (which is simply a sum over all distinguishable permutations of subscripts).

Define the generalized energy tensor $E^{[n]} := \int_{\mathbf{v}} f\mathbf{v}^n = \rho\langle\mathbf{v}^n\rangle$. Get the generalized conservative moment evolution equation

$$\partial_t E^{[n]} + \nabla \cdot E^{[n+1]} = \frac{q}{m} n \operatorname{Sym}\big(E^{[n-1]}\mathbf{E} + E^{[n]} \times \mathbf{B}\big) + \int_{\widetilde{\mathbf{v}}} \mathbf{v}^n C. \tag{2.1}$$

Setting $n = 0$ gives conservation of mass, setting $n = 1$ gives momentum evolution, and setting $n = 2$ gives energy tensor evolution, half of whose trace is energy evolution.

2.1 Primitive variables

Equation (2.1) is a coupled infinite system of evolution equations for moments of the Boltzmann equation. To provide finite closure we choose a maximum n and specify $E^{[n+1]}$ in terms of the lower moments. The problem of closure leads one naturally consider **primitive variables**. The closure relation should be

[1] Note that this notational convention is popular with physicists, whereas mathematicians often instead define $\langle\chi\rangle$ to be the simple velocity integral $\int_{\widetilde{\mathbf{v}}} \chi f$; I elect to use the notation of the physicists out of a predilection for molar densities.

invariant under change of inertial reference frame, so we are naturally lead to consider moments of the thermal speed $\mathbf{c} := \mathbf{v} - \mathbf{u}$, where $\mathbf{u} := \langle v \rangle$ is the bulk fluid velocity.

For $n \geq 2$ the primitive moments are defined by

$$P^{[n]} := \rho \langle \mathbf{c}^n \rangle,$$

which we will refer to as the n**th order generalized pressure**. For tensor orders 0 and 1 the primitive variables are defined to be ρ and $\mathbf{u}$. The primitive variable corresponding to the scalar energy is the scalar pressure, defined to be one third the trace of $\mathbb{P} := P^{[2]}$.

To relate primitive and conserved variables we observe that

$$\begin{aligned} v^n &= \mathrm{Sym}[v^n] = \mathrm{Sym}[(u+c)^n] = \mathrm{Sym} \sum_{j=0}^{n} \binom{n}{j} u^j c^{n-j}, \\ c^n &= \mathrm{Sym}[c^n] = \mathrm{Sym}[(v-u)^n] = \mathrm{Sym} \sum_{j=0}^{n} (-1)^j \binom{n}{j} u^j v^{n-j}. \end{aligned}$$

Primitive and conserved variables are thus related by

$$\begin{aligned} E^{[n]} &= \mathrm{Sym} \sum_{j=0}^{n} \binom{n}{j} \mathbf{u}^j P^{[n-j]} &&= P^{[n]} + \mathrm{Sym} \sum_{j=1}^{n-2} \binom{n}{j} \mathbf{u}^j P^{[n-j]} + \rho \mathbf{u}^n, \\ P^{[n]} &= \mathrm{Sym} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \mathbf{u}^j E^{[n-j]} &&= E^{[n]} + \mathrm{Sym} \sum_{j=1}^{n-2} (-1)^j \binom{n}{j} \mathbf{u}^j E^{[n-j]} + (-1)^n (1-n) \rho \mathbf{u}^n. \end{aligned}$$

Observe that for $n = 1$ and half the trace for $n = 2$ these formulas reduce to the familiar relations

$$\begin{aligned} \mathbf{M} &:= \rho \mathbf{u}, & \mathcal{E} &= \frac{3}{2} p + \frac{1}{2} \rho u^2, \\ \mathbf{u} &= \mathbf{M}/\rho, & p &= \frac{2}{3} \mathcal{E} - \frac{1}{3} \rho u^2, \end{aligned}$$

where $\mathbf{M}$ is the momentum, $\mathcal{E} := \frac{1}{2} \rho \langle v^2 \rangle$ is the energy, and $p := \frac{1}{3} \rho \langle c^2 \rangle$ is the pressure. More generally, we have:

$$\begin{aligned} E^{[0]} &= \rho, & P^{[0]} &= \rho, \\ E^{[1]} &= \rho \mathbf{u}, & P^{[1]} &= \mathbf{0}, \\ E^{[2]} &= \rho \mathbf{u}^2 + P^{[2]}, & P^{[2]} &= E^{[2]} - \rho \mathbf{u}^2, \\ E^{[3]} &= \rho \mathbf{u}^3 + \mathrm{Sym}\big(3 \mathbf{u} P^{[2]}\big) + P^{[3]}, & P^{[3]} &= E^{[3]} - \mathrm{Sym}(3 \mathbf{u} E^{[2]}) + 2 \rho \mathbf{u}^3, \\ E^{[4]} &= \rho \mathbf{u}^4 + \mathrm{Sym}\big(6 \mathbf{u}^2 P^{[2]} + 4 \mathbf{u} P^{[3]}\big) + P^{[4]}, & P^{[4]} &= E^{[4]} - \mathrm{Sym}\big(4 \mathbf{u} E^{[3]} - 6 \mathbf{u}^2 E^{[2]}\big) - 3 \rho \mathbf{u}^4, \\ E^{[5]} &= \rho \mathbf{u}^5 + \mathrm{Sym}\big(10 \mathbf{u}^3 P^{[2]} + 10 \mathbf{u}^2 P^{[3]} + 5 \mathbf{u} P^{[4]}\big) + P^{[5]}, & P^{[5]} &= E^{[5]} - \mathrm{Sym}\big(5 \mathbf{u} E^{[4]} - 10 \mathbf{u}^2 E^{[3]} + 10 \mathbf{u}^3 E^{[2]}\big) + 4 \rho \mathbf{u}^5 \\ &\vdots & &\vdots \end{aligned} \tag{2.2}$$

2.2 Temperature

The temperature is defined to be twice the average particle energy in a given direction (averaged over all three directions) in the reference frame of bulk flow: $T := \frac{1}{3} m \langle c^2 \rangle = \frac{p}{n}$, where $n := \rho/m$ is the number density of the species. By analogy, we define the generalized temperature tensor

$$T^{[n]} := P^{[n]}/n = m\langle \mathbf{c}^n \rangle.$$

Temperature is useful in positing closure relations; for example, one may posit that the heat flux is an isotropic linear function of the temperature gradient.

2.3 Closure.

To close a system of moment evolution equations up to $(n-1)$th order, we need to specify $E^{[n]}$ in terms of lower-order moments. We do so by positing a constitutive relation for the "heat flux tensor" $P^{[n]}$. Probably the simplest generic closure is truncation, i.e., assuming $P^{[n]} = 0$. Then (for $n \geq 2$)

$$E^{[n]} = E^{[n]} - P^{[n]} = -\operatorname{Sym} \sum_{j=1}^{n-2} (-1)^j \binom{n}{j} \mathbf{u}^j E^{[n-j]} + (-1)^n (n-1) \rho \mathbf{u}^n.$$

Specifically, we can use one of the closure approximations

$$\begin{aligned}
E^{[2]} &= \rho \mathbf{u}^2 && \text{(cold plasma)}, \\
E^{[3]} &= \operatorname{Sym}(3\mathbf{u}E^{[2]}) - 2\rho\mathbf{u}^3 && \text{(10-moment closure)} \\
E^{[4]} &= \operatorname{Sym}\big(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2 E^{[2]}\big) + 3\rho\mathbf{u}^4 && \text{(20-moment closure))} \\
E^{[5]} &= \operatorname{Sym}\big(5\mathbf{u}E^{[4]} - 10\mathbf{u}^2 E^{[3]} + 10\mathbf{u}^3 E^{[2]}\big) - 4\rho\mathbf{u}^5 && \text{(35-moment closure))} \\
&\vdots
\end{aligned}$$

to truncate the moment hierarchy

$$\begin{aligned}
\partial_t \rho + \nabla \cdot \mathbf{M} &= 0, \\
\partial_t \mathbf{M} + \nabla \cdot E^{[2]} &= \frac{q}{m} \operatorname{Sym}\big(\rho\mathbf{E} + \mathbf{M} \times \mathbf{B}\big), \\
\partial_t E^{[2]} + \nabla \cdot E^{[3]} &= 2\frac{q}{m} \operatorname{Sym}\big(\mathbf{M}\mathbf{E} + E^{[2]} \times \mathbf{B}\big), \\
\partial_t E^{[3]} + \nabla \cdot E^{[4]} &= 3\frac{q}{m} \operatorname{Sym}\big(E^{[2]}\mathbf{E} + E^{[3]} \times \mathbf{B}\big). \\
\partial_t E^{[4]} + \nabla \cdot E^{[5]} &= 4\frac{q}{m} \operatorname{Sym}\big(E^{[3]}\mathbf{E} + E^{[4]} \times \mathbf{B}\big). \\
&\vdots
\end{aligned}$$

A prohibitive problem with truncation closure is that it does not seem to give a hyperbolic system in case the highest moment has order greater than 2. The derivative of flux with respect to state has non-real eigenvalues, resulting in unbounded growth, i.e., an ill-posed system.

2.4 Contracted moments

In three spatial dimensions the number of independent entries in a totally symmetric nth order tensor in three spacial dimensions is $\binom{n+2}{2} = \frac{1}{2}(n+1)(n+2)$ and the number of moments up to nth order is $\binom{n+3}{3} =$

$\frac{1}{6}(n+1)(n+2)(n+3)$. (In *four* dimensions the number of independent entries in a totally symmetric nth order tensor is $\binom{n+3}{3}$ and the number of moments up to nth order is $\binom{n+4}{4} = \frac{1}{4!}(n+1)(n+2)(n+3)(n+4)$.)

n	$\binom{n+2}{2}$	$\binom{n+3}{3}$	$\binom{n+4}{4}$
0	1	1	1
1	3	4	5
2	6	10	15
3	10	20	35
4	15	35	70

To avoid the expense of evolving a high number of moments yet still retain higher-order information, one can replace the evolution equation for a higher moment with a contracted evolution equation for a contracted moment. To contract a tensor you set two indices equal and sum. For example, $\sum_i \alpha_{i,i,j}$ represent the contraction of the tensor α over its first two indices. Observe that the expectation of any power of particle velocity is a totally symmetric tensor, that any contraction of a totally symmetric tensor is a totally symmetric tensor, and that for a totally symmetric tensor it is irrelevant over which two indices you contract. We are thus motivated to define the trace tr of a totally symmetric tensor α to be its contraction over (any) two of its indices: $\operatorname{tr}\alpha := \mathbb{I} : \alpha = \alpha : \mathbb{I}$, where $\mathbb{I}$ is the identity tensor.

For example, the 14-moment system contracts the evolution equation for the 10 independent moments $\rho\langle\mathbf{vvv}\rangle = E^{[3]}$ to get an evolution equation for the 3 moments $\rho\langle\mathbf{vv}\cdot\mathbf{v}\rangle =: \operatorname{tr} E^{[3]}$, and twice contracts the evolution equation for the 15 independent moments $\rho\langle\mathbf{vvvv}\rangle = E^{[4]}$ to get an evolution equation for the scalar $\rho\langle\mathbf{v}\cdot\mathbf{vv}\cdot\mathbf{v}\rangle =: \operatorname{tr}\operatorname{tr} E^{[4]}$.

no. moments	evolved quantities					truncation
5	ρ,	$\rho\mathbf{u}$,	$\operatorname{tr} E^{[2]}$			$P^{[3]} = 0$
	1	+3	+1			
10	ρ,	$\rho\mathbf{u}$,	$E^{[2]}$			$P^{[3]} = 0$
	1	+3	+6			
14	ρ,	$\rho\mathbf{u}$,	$E^{[2]}$,	$\operatorname{tr} E^{[3]}$,	$\operatorname{tr}\operatorname{tr} E^{[4]}$	$P^{[5]} = 0$
	1	+3	+6	+3	+1	
21	ρ,	$\rho\mathbf{u}$,	$E^{[2]}$,	$E^{[3]}$,	$\operatorname{tr}\operatorname{tr} E^{[4]}$	$P^{[5]} = 0$
	1	+3	+6	+10	+1	
26	ρ,	$\rho\mathbf{u}$,	$E^{[2]}$,	$E^{[3]}$,	$\operatorname{tr} E^{[4]}$	$P^{[5]} = 0$
	1	+3	+6	+10	+6	
35	ρ,	$\rho\mathbf{u}$,	$E^{[2]}$,	$E^{[3]}$,	$E^{[4]}$	$P^{[5]} = 0$
	1	+3	+6	+10	+15	

Replacing a moment with a contracted moment poses a closure problem. To provide for closure one posits that the uncontracted moment $P^{[n]}$ is a linear isotropic function of its contracted moment.

In the case of the 5-moment system, positing that the pressure tensor is an isotropic linear function of a scalar gives the constitutive relation

$$P^{[2]} = \mathbb{I}p = \mathbb{I}\operatorname{tr} P^{[2]}/3. \tag{2.3}$$

In the case of the 14-moment system, this leads to the constitutive relations

$$P^{[3]}_{ijk} = \frac{1}{5}\sum_m \left(\delta_{ij}P^{[3]}_{kmm} + \delta_{ik}P^{[3]}_{jmm} + \delta_{jk}P^{[3]}_{imm}\right) = \frac{3}{5}\operatorname{Sym}(\mathbb{I}\otimes \operatorname{tr} P^{[3]})$$

and

$$P^{[4]}_{ijkl} = \frac{1}{15}\sum_m\sum_n P^{[4]}_{mmnn}\Big(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il}\Big) = \frac{3}{15}\,\mathrm{tr}\ \mathrm{tr}\, P^{[4]}\,\mathrm{Sym}(\mathbb{I}\otimes\mathbb{I}), \tag{2.4}$$

and in the case of the 26-moment system one gets the constitutive relation [NEED TO FINISH THIS]

where $\mathbb{I}$ is the identity tensor, δ_{ij} denotes Kronecker delta, and Sym is the map which takes a tensor and returns its symmetric part (obtained e.g. by averaging over all permutations of subscripts). The truncation closure remains $P^{[5]} = 0$.

2.5 10-moment system

In conserved variables the ten-moment system is

$$\begin{aligned}
\partial_t\rho + {}& \nabla\cdot\mathbf{M} = 0,\\
\partial_t\mathbf{M} + \nabla\cdot E^{[2]} &= \frac{q}{m}\,\mathrm{Sym}\big(\rho\mathbf{E} + \mathbf{M}\times\mathbf{B}\big),\\
\partial_t E^{[2]} + \nabla\cdot E^{[3]} &= 2\frac{q}{m}\,\mathrm{Sym}\big(\mathbf{M}\mathbf{E} + E^{[2]}\times\mathbf{B}\big),\\
E^{[3]} &= \mathrm{Sym}(3\mathbf{u}E^{[2]}) - 2\rho\mathbf{u}^3.
\end{aligned}$$

That is,

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbb{P}) = \frac{q}{m}\rho(\mathbf{E} + \mathbf{u}\times\mathbf{B}),\\
&\partial_t(\rho\mathbf{u}\mathbf{u} + \mathbb{P}) + \nabla\cdot\big(\rho\mathbf{u}\mathbf{u}\mathbf{u} + 3\,\mathrm{Sym}(\mathbf{u}\mathbb{P})\big) = \frac{q}{m}2\,\mathrm{Sym}\big(\rho\mathbf{u}\mathbf{E} + (\mathbb{P} + \rho\mathbf{u}\mathbf{u})\times\mathbf{B}\big).
\end{aligned}$$

2.6 5-moment system

The 5-moment system replaces the evolution equation for the second moment with half its trace and uses the isotropic pressure constitutive relation (2.3),

$$P^{[2]} = \mathbb{I}p = \mathbb{I}\,\mathrm{tr}\,P^{[2]}/3.$$

To obtain a corresponding constitutive relation in conserved variables for $E^{[2]}$, recall that (2.2) expresses $P^{[2]}$ in terms of conserved variables:

$$P^{[2]} = E^{[2]} - \rho\mathbf{u}\mathbf{u}. \tag{2.5}$$

(We remark that taking half the trace of this equation gives the familiar relation $\mathcal{E} = \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{3}{2}p$, where we define the scalar energy by $\mathcal{E} := \frac{1}{2}\,\mathrm{tr}\,E^{[2]}$.) Substituting this equation and its trace into the primitive-variables constitutive relation gives the conservative-variables constitutive relation

$$\begin{aligned}
E^{[2]} &= \rho\mathbf{u}\mathbf{u} + \mathbb{I}p\\
&= \rho\mathbf{u}\mathbf{u} + \mathbb{I}\,\mathrm{tr}\left(\frac{1}{3}E^{[2]} - \frac{1}{3}\rho\mathbf{u}\mathbf{u}\right)\\
&= \rho\mathbf{u}\mathbf{u} + \mathbb{I}\Big(\frac{2}{3}\mathcal{E} - \frac{1}{3}\rho|\mathbf{u}|^2\Big).
\end{aligned}$$

Substituting this into the momentum evolution equation and taking the trace of the energy equation gives a closed system. Since

$$\begin{aligned}\frac{1}{2}\operatorname{tr} E^{[3]} &= \frac{1}{2}\operatorname{tr}\big[\operatorname{Sym}(3\mathbf{u}E^{[2]}) - 2\rho\mathbf{u}^3\big],\\ &= E^{[2]}\cdot\mathbf{u} + \mathcal{E}\mathbf{u} - \rho|\mathbf{u}|^2\mathbf{u},\\ &= \Big(\frac{5}{3}\mathcal{E} - \frac{1}{3}\rho|\mathbf{u}|^2\Big)\mathbf{u},\end{aligned}$$

the 5-moment system in conserved variables is

$$\begin{aligned}\partial_t\rho + \nabla\cdot\mathbf{M} &= 0,\\ \partial_t\mathbf{M} + \nabla\cdot\rho\mathbf{u}\mathbf{u} + \nabla\Big(\frac{2}{3}\mathcal{E} - \frac{1}{3}\rho|\mathbf{u}|^2\Big) &= \frac{q}{m}\operatorname{Sym}\big(\rho\mathbf{E} + \mathbf{M}\times\mathbf{B}\big),\\ \partial_t\mathcal{E} + \nabla\cdot\Big(\frac{5}{3}\mathcal{E}\mathbf{u} - \frac{1}{3}\rho|\mathbf{u}|^2\mathbf{u}\Big) &= \frac{q}{m}\mathbf{M}\cdot\mathbf{E}.\end{aligned}$$

2.7 21-moment (and 14-moment) closure for $E^{[4]}$

The 21-moment system replaces the evolution equation for $E^{[4]}$ with the trace of its trace.

Equation (2.2) gives $E^{[4]}$ in terms of $P^{[4]}$ and lower conserved moments:

$$E^{[4]} = P^{[4]} + \operatorname{Sym}\big(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}\big) + 3\rho\mathbf{u}^4,$$

But $P^{[4]}$ is given from twice its trace by (2.4),

$$P^{[4]}_{ijkl} = \frac{3}{15}\operatorname{tr}\operatorname{tr} P^{[4]}\operatorname{Sym}(\mathbb{I}\otimes\mathbb{I}).$$

Twice taking the trace of (2.2) for $P^{[4]}$ in terms of $E^{[4]}$ and substituting into (2.4) gives a constitutive relation for $E^{[4]}$ in terms of its trace and lower moments,

$$E^{[4]} = \operatorname{tr}\operatorname{tr}\big(E^{[4]} - \operatorname{Sym}\big(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}\big) - 3\rho\mathbf{u}^4\big)\frac{3}{15}\operatorname{Sym}(\mathbb{I}\otimes\mathbb{I}) + \operatorname{Sym}\big(4\mathbf{u}E^{[3]} - 6\mathbf{u}^2E^{[2]}\big) + 3\rho\mathbf{u}^4.$$

2.8 14-moment closure for $E^{[3]}$

The 14-moment system is obtained from the 21-moment system by taking the trace of the evolution equation for $E^{[3]}$. Equation (2.2) gives $E^{[3]}$ in terms of $P^{[3]}$ and lower conserved moments:

$$E^{[3]} = P^{[3]} + \operatorname{Sym}\big(3\mathbf{u}E^{[2]}\big) - 2\rho\mathbf{u}^4,$$

But $P^{[3]}$ is given from its trace by (2.4),

$$P^{[3]}_{ijk} = \frac{3}{5}\operatorname{Sym}(\mathbb{I}\otimes\operatorname{tr} P^{[3]}).$$

Taking the trace of (2.2) for $P^{[3]}$ in terms of $E^{[3]}$ and substituting into (2.4) gives a constitutive relation for $E^{[3]}$ in terms of its trace and lower moments:

$$E^{[3]} = \frac{3}{5}\operatorname{Sym}\big(\mathbb{I}\otimes\operatorname{tr}\big(E^{[3]} - \operatorname{Sym}\big(3\mathbf{u}E^{[2]}\big) + 2\rho\mathbf{u}^4\big)\big) + \operatorname{Sym}(3\mathbf{u}E^{[2]}) - 2\rho\mathbf{u}^4.$$

3 Evolution of conserved moments

3.1 Momentum evolution

Let $\chi = \widetilde{\mathbf{v}}$. Define the average velocity $\mathbf{u} := \langle \mathbf{v} \rangle$ and the average proper velocity $\widetilde{\mathbf{u}} := \langle \widetilde{\mathbf{v}} \rangle$. Define the thermal velocity $\mathbf{c} := \mathbf{v} - \langle \mathbf{v} \rangle$ and the thermal proper velocity $\widetilde{\mathbf{c}} := \widetilde{\mathbf{v}} - \langle \widetilde{\mathbf{v}} \rangle$. The momentum is $\mathbf{M} := \rho \widetilde{\mathbf{u}}$. Then the velocity moment evolution becomes the the balance law for momentum,

$$\partial_t(\rho \widetilde{\mathbf{u}}) + \nabla \cdot (\rho \mathbf{u} \widetilde{\mathbf{u}} + \widetilde{\mathbb{P}}) = \frac{q}{m} \rho (\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \int_{\widetilde{\mathbf{v}}} \widetilde{\mathbf{v}} C,$$

where $\widetilde{\mathbb{P}} := \rho \langle \mathbf{c} \widetilde{\mathbf{c}} \rangle$ is the pressure tensor.

3.2 "Energy" tensor evolution

[From here on results hold only for the non-relativistic domain.] Let $\chi = \mathbf{vv}$. So $\langle \chi \rangle = \mathbf{uu} + \langle \mathbf{cc} \rangle$. Define the pressure tensor $\mathbb{P} := \rho \langle \mathbf{cc} \rangle$ and the "energy tensor" $\mathbb{E} := \rho \langle \mathbf{vv} \rangle$ (whose trace is twice the gas-dynamic energy). So $\mathbb{E} := \rho \mathbf{uu} + \mathbb{P}$, where $\rho \mathbf{uu}$ is the "kinetic energy tensor".

We calculate the terms of the velocity moment evolution equation.

$$\rho \langle \mathbf{vvv} \rangle = \rho(\mathbf{uuu} + \langle \mathbf{ccu} \rangle + \langle \mathbf{cuc} \rangle + \langle \mathbf{ucc} \rangle + \langle \mathbf{ccc} \rangle) = \rho(\mathbf{uuu} + 3\,\mathrm{Sym}\langle \mathbf{ucc} \rangle + \langle \mathbf{ccc} \rangle))$$

and

$$\begin{aligned} &\rho \langle \mathbf{a} \cdot \nabla_{\mathbf{v}} \cdot \mathbf{vv} \rangle = \rho \langle \mathbf{av} + \mathbf{va} \rangle = 2\rho\,\mathrm{Sym}\langle \mathbf{av} \rangle = 2\rho\,\mathrm{Sym}(\langle \mathbf{a} \rangle \mathbf{u} + \langle \mathbf{ac} \rangle) \\ &= 2\frac{q}{m}\rho\,\mathrm{Sym}((\mathbf{E} + \mathbf{u} \times \mathbf{B})\mathbf{u} + \langle \mathbf{c} \times \mathbf{Bc} \rangle) = 2\frac{q}{m}\,\mathrm{Sym}(\rho \mathbf{uE} + (\rho \mathbf{uu} + \mathbb{P}) \times \mathbf{B}). \end{aligned}$$

The velocity moment evolution equation becomes the energy tensor evolution equation

$$\partial_t(\rho \mathbf{uu} + \mathbb{P}) + \nabla \cdot \big(\rho \mathbf{uuu} + 3\,\mathrm{Sym}(\mathbf{u}\mathbb{P}) + \mathbb{P}^{[3]}\big) = \frac{q}{m} 2\,\mathrm{Sym}\big(\rho \mathbf{uE} + (\mathbb{P} + \rho \mathbf{uu}) \times \mathbf{B}\big) + \int_{\mathbf{v}} \mathbf{vv} C. \quad (3.1)$$

Taking half the trace of this gives the energy evolution equation,

$$\partial_t \mathcal{E} + \nabla \cdot \big(\mathbf{u}\mathcal{E} + \mathbf{u} \cdot \mathbb{P}) + \rho \langle \mathbf{c} c^2 \rangle\big) = \frac{q}{m} \rho \mathbf{u} \cdot \mathbf{E} + \int_{\mathbf{v}} C v^2/2. \quad (3.2)$$

4 Evolution of primitive moments

4.1 Evolution of generalized moment

Let $\chi(t, \mathbf{x}, \mathbf{v})$ be a generic generalized moment. (We will later impose $\chi(\mathbf{c})$. Note that $\mathbf{c}(t, \mathbf{x}, \mathbf{v}) = \mathbf{v} - \mathbf{u}(t, \mathbf{x})$.) Multiply the Boltzmann equation by χ and integrate by parts. Get the generic moment evolution equation

$$\partial_t(\rho \langle \chi \rangle) + \nabla \cdot (\rho \langle \mathbf{v} \chi \rangle) = \rho \langle (d_t^{\mathbf{v}} + \mathbf{a} \cdot \nabla_{\mathbf{v}}) \chi \rangle + \int_{\mathbf{v}} \chi C,$$

where $d_t^{\mathbf{v}} := \partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}}$.

4.2 Evolution of generic thermal velocity moment

Impose $\chi(\mathbf{c})$, where $\mathbf{c} = \mathbf{v} - \mathbf{u}(t, \mathbf{x})$. So $\nabla_{\mathbf{v}} = \nabla_{\mathbf{c}}$; also, $d_t^{\mathbf{v}}\chi = (d_t^{\mathbf{v}}\mathbf{c}) \cdot \nabla_{\mathbf{c}}\chi = -(d_t^{\mathbf{v}}\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi$. So the generic moment evolution equation becomes

$$\partial_t(\rho\langle\chi\rangle) + \nabla \cdot (\rho\mathbf{u}\langle\chi\rangle) + \nabla \cdot (\rho\langle\mathbf{c}\chi\rangle) = \rho\langle(\mathbf{a} - d_t^{\mathbf{v}}\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi\rangle + \int_{\mathbf{v}} \chi C.$$

But momentum conservation says that $\langle\mathbf{a}\rangle - d_t^{\mathbf{u}}\mathbf{u} = (\nabla \cdot \mathbb{P} - \mathbf{R})/\rho$, where $\mathbf{R} = \int_{\mathbf{v}} \mathbf{v}C$ is collisional resistance. So $\mathbf{a} - (d_t^{\mathbf{v}}\mathbf{u}) = \mathbf{a}' + \langle\mathbf{a}\rangle - d_t^{\mathbf{u}}\mathbf{u} - \mathbf{c} \cdot \nabla\mathbf{u} = (\nabla \cdot \mathbb{P} - \mathbf{R})/\rho + \mathbf{a}' - \mathbf{c} \cdot \nabla\mathbf{u}$ where $\mathbf{a}' := \mathbf{a} - \langle\mathbf{a}\rangle = \frac{q}{m}\mathbf{c} \times \mathbf{B}$. So the generic thermal velocity moment evolution equation is

$$\partial_t(\rho\langle\chi\rangle) + \nabla \cdot (\rho\mathbf{u}\langle\chi\rangle) + \nabla \cdot (\rho\langle\mathbf{c}\chi\rangle) = (\nabla \cdot \mathbb{P} - \mathbf{R}) \cdot \langle\nabla_{\mathbf{c}}\chi\rangle + \rho\langle(\mathbf{a}' - \mathbf{c} \cdot \nabla\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi\rangle + \int_{\mathbf{v}} \chi C. \tag{4.1}$$

4.3 Evolution of generalized pressure tensor

Choose $\chi = \chi^{[n]} := \prod_{i=1}^{n} \mathbf{c}$. Then (4.1) gives an evolution equation for the generalized pressure $\mathbb{P}^{[n]} := \rho\langle\chi^{[n]}\rangle$. We seek to express the rest of the equation in terms of generalized pressures. Note that $\chi = \mathrm{Sym}(\chi)$ and $\mathbb{P}^{[n]} = \mathrm{Sym}(\mathbb{P}^{[n]})$. For a generic $\underline{\alpha}$,

$$\underline{\alpha} \cdot \nabla_{\mathbf{c}}\chi^{[n]} = \sum_j \alpha_j \partial_{c_j} \mathrm{Sym}(\chi^{[n]}) = n\,\mathrm{Sym}(\underline{\alpha}\chi^{[n-1]}).$$

So

$$\begin{aligned}\rho\langle(\mathbf{a}' - \mathbf{c} \cdot \nabla\mathbf{u}) \cdot \nabla_{\mathbf{c}}\chi^{[n]}\rangle &= n\rho\,\mathrm{Sym}\langle(\mathbf{a}' - \mathbf{c} \cdot \nabla\mathbf{u})\chi^{[n-1]}\rangle \\ &= n\,\mathrm{Sym}\big(\frac{q}{m}\mathbb{P}^{[n]} \times \mathbf{B} - \mathbb{P}^{[n]} \cdot \nabla\mathbf{u}\big)\end{aligned}$$

The generic thermal velocity moment evolution equation becomes the following generalized pressure tensor evolution equation,

$$\eth_t(\mathbb{P}^{[n]}) + \nabla \cdot (\mathbb{P}^{[n+1]}) + n\,\mathrm{Sym}\Big(\mathbb{P}^{[n-1]}(\mathbf{R} - \nabla \cdot \mathbb{P}^{[2]})/\rho + \mathbb{P}^{[n]} \cdot \nabla\mathbf{u}\Big) = n\,\mathrm{Sym}\big(\frac{q}{m}\mathbb{P}^{[n]} \times \mathbf{B}\big) + \int_{\mathbf{c}} C \prod_{i=1}^{n} \mathbf{c}.$$

4.4 Evolution of pressure tensor

In case $n = 2$, write $\mathbb{P} := \mathbb{P}^{[2]} = \rho\langle\mathbf{cc}\rangle$. $\mathbb{P}^{[1]} = \rho\langle\mathbf{c}\rangle = 0$. So the pressure tensor evolution equation becomes

$$\eth_t(\mathbb{P}) + \nabla \cdot (\mathbb{P}^{[3]}) + 2\,\mathrm{Sym}\big(\mathbb{P} \cdot \nabla\mathbf{u}\big) = 2\,\mathrm{Sym}\big(\frac{q}{m}\mathbb{P} \times \mathbf{B}\big) + \int_{\mathbf{c}} C\mathbf{cc},$$

i.e.,

$$\partial_t(\mathbb{P}) + \nabla \cdot \big(\mathbb{P}^{[3]} + 3\,\mathrm{Sym}(\mathbf{u}\mathbb{P})\big) - 2\,\mathrm{Sym}(\mathbf{u}\nabla \cdot \mathbb{P}) = 2\,\mathrm{Sym}\big(\frac{q}{m}\mathbb{P} \times \mathbf{B}\big) + \int_{\mathbf{c}} C\mathbf{cc}.$$

Subtracting this from the evolution equation (3.2) for the energy tensor gives a kinetic energy tensor evolution equation,

$$\partial_t(\rho\mathbf{uu}) + \nabla \cdot \big(\rho\mathbf{uuu}\big) + 2\,\mathrm{Sym}(\mathbf{u}\nabla \cdot \mathbb{P}) = \frac{q}{m}2\,\mathrm{Sym}\big(\rho\mathbf{u}\mathbf{E} + (\rho\mathbf{uu}) \times \mathbf{B}\big) + \int_{\mathbf{v}} C(\mathbf{vv} - \mathbf{cc}),$$

which can be obtained by multiplying the momentum equation $\rho d_t \mathbf{u} = \rho \frac{q}{m}\big(\mathbf{E} + \mathbf{u} \times \mathbf{B}\big)$ by $\mathbf{u}$ and taking the symmetric part. For closure we neglect $\mathbb{P}^{[3]}$ (which is zero if we assume that the pressure tensor is an anisotropic Gaussian) and assume that the collision operator C is zero. This will close the system, giving us the ten-moment collisionless plasma model.

4.5 Evolution of pressure

To get an evolution equation for the pressure we take the trace of the evolution equation for the pressure tensor.