Plasma notes: A derivation of Ohm's law and extended MHD starting from two-fluid equations

by E. Alec Johnson, Dec. 2007 – May 2008.

1 Introduction

This document derives MHD starting with a generic two-fluid system of evolution equations for mass density, momentum, energy, and electromagnetic field.

MHD models are characterized by their use of Ohm's law to specify the electric field. The following basic assumptions underlie MHD models:

1. neglect of displacement current,
2. quasineutrality,
3. neglect of second-order gas-dynamic terms that arise from interspecies drift velocities (my carefulness to avoid this assumption below leads to complicated source terms that are usually neglected), and
4. the typical gas-dynamic constitutive assumptions, e.g. closure relations for the stress tensor and heat flux.

2 Presentation of the two-fluid equations

Generic two-fluid equations for a two-species plasma are:

$$\partial_t \begin{bmatrix} \rho_i \\ \rho_e \\ \rho_i \mathbf{u}_i \\ \rho_e \mathbf{u}_e \\ \mathcal{E}_i \\ \mathcal{E}_e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho_i \mathbf{u}_i \\ \rho_e \mathbf{u}_e \\ \rho_i \mathbf{u}_i \mathbf{u}_i + \mathbb{P}_i \\ \rho_e \mathbf{u}_e \mathbf{u}_e + \mathbb{P}_e \\ \mathbf{u}_i \mathcal{E}_i + \mathbf{u}_i \cdot \mathbb{P}_i + \mathbf{q}_i \\ \mathbf{u}_e \mathcal{E}_e + \mathbf{u}_e \cdot \mathbb{P}_e + \mathbf{q}_e \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 0 \\ 0 \\ \sigma_i \mathbf{E} + \mathbf{J}_i \times \mathbf{B} \\ \sigma_e \mathbf{E} + \mathbf{J}_e \times \mathbf{B} \\ \mathbf{J}_i \cdot \mathbf{E} \\ \mathbf{J}_e \cdot \mathbf{E} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{R}_i \\ \mathbf{R}_e \\ \mathbf{R}_i \cdot \mathbf{u}_i + Q_{R,i} + Q_i \\ \mathbf{R}_e \cdot \mathbf{u}_e + Q_{R,e} + Q_e \end{bmatrix},$$

$$\partial_t \begin{bmatrix} (c\mathbf{B}) \\ \mathbf{E} \end{bmatrix} + c\nabla \times \begin{bmatrix} \mathbf{E} \\ -(c\mathbf{B}) \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 0 \\ -\mathbf{J}/\lambda^2 \end{bmatrix}, \text{ and } \nabla \cdot \begin{bmatrix} (c\mathbf{B}) \\ \mathbf{E} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 0 \\ \sigma/\lambda^2 \end{bmatrix}.$$

The variables are defined as follows: $\mathbf{B}$ is magnetic field, $\mathbf{E}$ is electric field, c is the speed of light, and i and e are ion and electron species indices; for species $s \in \{i, e\}$, q_s is particle charge, m_s is particle mass, n_s is particle number density, $\rho_s = m_s n_s$ is mass density, $\sigma_s = q_s n_s$ is charge density, $\mathbf{J}_s = \mathbf{u}_s \sigma_s$ is current density, $\mathbb{P}_s$ is the pressure tensor, $\mathcal{E}_s$ is gas-dynamic energy, $\mathbf{q}_s$ is the heat flux, $\mathbf{R}_i = -\mathbf{R}_e$ denotes the interspecies drag force on the ions, $Q_{R,s}$ denotes heating due to friction (drag), and $Q_i = -Q_e$ denotes the interspecies thermal heat transfer to the ions. The scaling of the interspecies drag force and heat transfer are not considered here. We assume singly charged ions: $q_i = -q_e =: e$. In case one wishes to view all equations using standard dimensions, take $r = 1$ and take $\lambda^2 = \epsilon_0$, the permittivity of free space. The parameters r and λ are present so that this derivation will apply also for a nondimensionalization of these equations. For the nondimensionalized case, $r = \frac{m_0 u_0}{q_0 B_0 x_0}$ is the nondimensionalized gyroradius of a typical ion in the presence of a typical magnetic field over a typical length scale, and $\lambda^2 = \frac{\epsilon_0 B_0^2}{n_0 m_0}$ defines λ, the ratio of the Debye length to the gyroradius. The Debye length is the charge-shielding distance for an ion in a plasma of typical density and temperature. We will generically denote $\mu_0^{-1} := c^2\lambda^2$ (and likewise redefine $\epsilon_0 := r\lambda^2$). The parameters that determine the behavior of the plasma are c, λ, r, and the mass ratio $m := m_i/m_e \approxeq 1836$ (hidden in the definition of net current, manifest in Ohm's law below).

In the collisionless ideal two-fluid model, we neglect the heat fluxes $\mathbf{q}_s$ and the transfer terms, i.e., $\mathbf{R}_i \approx 0 \approx \mathbf{R}_e$, $Q_i \approx 0 \approx Q_e$, and $Q_{R,i} \approx 0 \approx Q_{R,e}$. We also assume that pressure is scalar, i.e., the pressure tensor is isotropic. Recall that for point particle species, which only have the 3 translational degrees of freedom, the thermal energy is one half the trace of the pressure tensor. This yields the ideal gas constitutive relation which serves to specify the pressure: $\mathcal{E}_s = (3/2)p_s + \rho_s u_s^2/2$.

More general constitutive relations could involve assuming that the heat flux is proportional to the temperature gradient, that the interspecies drag force is proportional to the species densities and relative velocities, that the interspecies heat transfer is proportional to the species densities and difference in temperature, that there are distinct scalar pressures parallel and perpendicular to the magnetic field, or that the pressure tensor has a viscous component that depends linearly and isotropically on the velocity gradient.

3 Quasineutrality and the one-fluid model

Summing the density, momentum, and energy equations over both species yields net fluid balance laws, and summing the constitutive relations for the pressure yields a net constitutive relation:

$$\begin{aligned}
&\partial_t \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \mathcal{E} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + \mathbb{P}^{\mathrm{d}} + \mathbb{P} \\ \mathbf{u}\mathcal{E} + \mathbf{u}\cdot\mathbb{P} + \mathbf{q}^{\mathrm{d}} + \mathbf{q} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 0 \\ \sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} \\ \mathbf{J}\cdot\mathbf{E} \end{bmatrix} \quad \text{and} \\
&\mathcal{E} = (3/2)p + \rho u^2/2,
\end{aligned}$$

where the density ρ, momentum $\rho\mathbf{u}$, energy $\mathcal{E}$, pressure tensor $\mathbb{P}$, charge density σ, current density $\mathbf{J}$, and heat flux $\mathbf{q}$ are the sum of their respective ion and electron quantities. (So $(\rho\mathbf{u}) := \rho_i\mathbf{u}_i + \rho_e\mathbf{u}_e$ defines the net velocity $\mathbf{u}$.) To deal with the nonlinear flux terms in the momentum and energy equations, we here have defined the diffusion velocities $\mathbf{w}_s$ relative to the fluid velocity by $\mathbf{u}_i = \mathbf{u} + \mathbf{w}_i$ and $\mathbf{u}_e = \mathbf{u} + \mathbf{w}_e$. Here the diffusion pressure and diffusion heat flux are defined by

$$\begin{aligned}
\mathbb{P}^{\mathrm{d}} &:= \rho_i\mathbf{w}_i\mathbf{w}_i + \rho_e\mathbf{w}_e\mathbf{w}_e, \\
\mathbf{q}^{\mathrm{d}} &:= \sum_s (\mathbf{w}_s\mathcal{E}_s + \mathbf{w}_s\cdot\mathbb{P}_s).
\end{aligned}$$

(We use the word "diffusion" here to refer to two fluid species diffusing through one another, rather than thermal particle motion diffusing their properties through a medium. In terms of partial differential equations, I'm thinking that relative motion of fluid species is dispersive or diffusive depending on the quantities being convected, but I'm not completely sure.)

The fundamental assumption of one-fluid models is *quasineutrality*: $\sigma \approx 0$. This assumption is appropriate on space scales greater than the Debye length, which is the length scale over which electric fields are screened out by a redistribution of the electrons. Quasineutrality means that ion and electron particle densities are approximately equal: $n_i \approx n_e \approx n := (n_i + n_e)/2$. So the net current is the same in any reference frame. Since we can also say that the net momentum is zero in a reference frame moving with the fluid, we have the system:

$$\begin{aligned} \mathbf{J}/(en) &= \mathbf{w}_i - \mathbf{w}_e, \\ 0 &= m_i \mathbf{w}_i + m_e \mathbf{w}_e. \end{aligned}$$

Solving for diffusion velocities yields

$$\begin{aligned} \mathbf{w}_i &= \frac{m_e}{m_i + m_e} \frac{\mathbf{J}}{en} = \frac{\tilde{m}_e \mathbf{J}}{\rho}, \quad \text{where } \tilde{m}_e := \frac{m_e}{e}, \text{ and} \\ \mathbf{w}_e &= \frac{-m_i}{m_i + m_e} \frac{\mathbf{J}}{en} = -\frac{\tilde{m}_i \mathbf{J}}{\rho}, \text{ where } \tilde{m}_i := \frac{m_i}{e}. \end{aligned} \tag{3.1}$$

So the assumption of quasineutrality allows us to express ρ_s and $\mathbf{w}_s$ (equivalently, $\mathbf{u}_s$) in terms of one-fluid quantities. One typically assumes the constitutive relation $\mathbf{R}_i = \nu n^2 (\mathbf{w}_e - \mathbf{w}_i)$, where $\nu \geq 0$ is a proportionality constant. So we have left to eliminate the species quantities $\mathbb{P}_s$ and $\mathcal{E}_s$. Recall that $2\mathcal{E}_s = \operatorname{trace} \mathbb{P}_s + \rho_s u_s^2$, and consequently $2\mathcal{E} = \operatorname{trace} \mathbb{P} + \rho u^2 + \rho_i w_i^2 + \rho_e w_e^2$. (Nontranslational degrees of freedom require the addition of another thermal energy term on the right hand side in addition to the trace of the pressure tensor.) So the closure problem reduces to finding a constitutive relation for the ion and electron pressure tensors. The simplest approach is just to neglect all the terms above involving diffusion velocity. The second simplest approach is to assume that the electron and ion temperature are in a fixed ratio (which, for the ideal gas law $p_s = n_s(kT_s)$, under the quasineutrality assumption $n_i = n_e$, means that the pressures are in the same fixed ratio, e.g. 1.) A more complicated approach is to retain separate evolution equations for the energy, temperature, or pressure of each individual species. (This works because for a quasineutral fluid n, ρ_s, $\mathbf{u}_s$, and $\mathbf{J}_s = n q_s \mathbf{u}_s$ are all known in terms of ρ, $\mathbf{u}$, and $\mathbf{J}$.)

We now determine the "diffusion" quantities explicitly in terms of one-fluid variables.

The diffusion pressure tensor is

$$\mathbb{P}^{\mathrm{d}} = \rho_i \mathbf{w}_i \mathbf{w}_i + \rho_e \mathbf{w}_e \mathbf{w}_e = \frac{\tilde{m}_i}{\tilde{m}_i + \tilde{m}_e} \frac{\tilde{m}_e^2 \mathbf{J}\mathbf{J}}{\rho} + \frac{\tilde{m}_e}{\tilde{m}_i + \tilde{m}_e} \frac{\tilde{m}_i^2 \mathbf{J}\mathbf{J}}{\rho} = \tilde{m}_i \tilde{m}_e \frac{\mathbf{J}\mathbf{J}}{\rho}. \tag{3.2}$$

Using the same kind of derivation (or using that the diffusion kinetic energy is half the trace of the diffusion pressure tensor) gives the "diffusion" scalar pressure,

$$\frac{3}{2} p^{\mathrm{d}} := \frac{1}{2} [\rho_i w_i^2 + \rho_e w_e^2] = \tilde{m}_i \tilde{m}_e \frac{J^2}{2\rho}. \tag{3.3}$$

The "diffusive" heat flux tensor is

$$\begin{aligned} \mathbf{q}^{\mathrm{d}} &= \sum_s (\mathbf{w}_s \mathcal{E}_s + \mathbf{w}_s \cdot \mathbb{P}_s) = \sum_s (\mathbf{w}_s (\frac{1}{2} \rho_s u_s^2 + \frac{3}{2} p_s + p_s) - \mathbf{w}_s \cdot \mathbb{T}_s) \\ &= \tilde{m} \mathbf{J} (u_i^2 - u_e^2)/2 + \frac{5}{2} \frac{\mathbf{J}}{\rho} (\tilde{m}_e p_i - \tilde{m}_i p_e) - \frac{\mathbf{J}}{\rho} \cdot (\tilde{m}_e \mathbb{T}_i - \tilde{m}_i \mathbb{T}_e), \end{aligned}$$

where $\tilde{m} := \frac{\tilde{m}_e \tilde{m}_i}{\tilde{m}_e + \tilde{m}_i}$ is the *reduced mass.* But

$$\begin{aligned}
\tilde{m}\mathbf{J}(u_i^2 - u_e^2)/2 &= \tilde{m}\mathbf{J}(\mathbf{u}_i - \mathbf{u}_e) \cdot \frac{\mathbf{u}_i + \mathbf{u}_e}{2} \\
&= \tilde{m}\mathbf{J}(\mathbf{w}_i - \mathbf{w}_e) \cdot \left(\mathbf{u} + \frac{\mathbf{w}_i + \mathbf{w}_e}{2}\right) \\
&= \tilde{m}_i \tilde{m}_e \frac{\mathbf{J}\mathbf{J}}{\rho} \cdot \left(\mathbf{u} + \frac{\mathbf{w}_i + \mathbf{w}_e}{2}\right) \\
&= \tilde{m}_i \tilde{m}_e \frac{\mathbf{J}\mathbf{J}}{\rho} \cdot \left(\mathbf{u} + \frac{\tilde{m}_e - \tilde{m}_i}{2} \frac{\mathbf{J}}{\rho}\right).
\end{aligned}$$

Notice that we get the diffusion pressure dot the average velocity. So

$$\mathbf{q}^{\mathrm{d}} = \tilde{m}_i \tilde{m}_e \frac{\mathbf{J}\mathbf{J}}{\rho} \cdot \left(\mathbf{u} + \frac{\tilde{m}_e - \tilde{m}_i}{2} \frac{\mathbf{J}}{\rho}\right) + \frac{5}{2} \frac{\mathbf{J}}{\rho} (\tilde{m}_e p_i - \tilde{m}_i p_e) - \frac{\mathbf{J}}{\rho} \cdot (\tilde{m}_e \mathbb{T}_i - \tilde{m}_i \mathbb{T}_e).$$

Assuming that the drag force is proportional to the interspecies velocity gives:

$$\mathbf{R}_i = -\mathbf{R}_e = \nu n^2 (\mathbf{w}_e - \mathbf{w}_i) = -\nu n^2 (\tilde{m}_e + \tilde{m}_i) \mathbf{J}/\rho = -\nu n \mathbf{J}/e.$$

3.1 Interspecies diffusive energy exchange

Quasineutrality also allows us to derive some simple expressions for interspecies energy exchange. (These effects cancel when the energy equations are summed for both species.) The rate of work due to the drag force is:

$$\mathbf{R}_i \cdot (\mathbf{w}_i - \mathbf{w}_e) = -\nu n \frac{\mathbf{J}}{e} \cdot \frac{\mathbf{J}}{en} = \frac{-\nu}{e^2} J^2.$$

This represents energy dissipated in the form of frictional heating. As in [**?**], p45, we posit that the frictional heating is distributed inversely as the particle masses, based on the observation that in a grazing collision conservation of momentum says that the perturbation in kinetic energy is distributed inversely as the particle masses.

4 The "magnetostatic" assumption

The *magnetohydrodynamic (MHD)* plasma model adds to the one-fluid model a second fundamental assumption (which I'll refer to as the *"magnetostatic assumption"*) which assumes that $\partial_t \mathbf{E}$ can be neglected in Ampere's law. This assumption removes light waves from the system, which maybe is appropriate on space scales greater than the plasma skin depth, which I think is the depth in a plasma to which electromagnetic radiation can penetrate [CHECK]. This magnetostatic assumption allows us to eliminate the current by expressing it in terms of the magnetic field.

4.1 Ohm's law

Since magnetostatics neglects $\partial_t \mathbf{E}$, the electric field in the source terms needs to be supplied by a constitutive relation called *Ohm's law.* Recall from elementary physics that Ohm's law specifies the electric field (equivalently, the voltage gradient) in a resistor induced by a given current. Similarly, Ohm's law in the plasma context specifies the electric field induced by the state variables of the plasma. Ohm's law is simply the current balance equation solved for the electric field and expressed in terms of of one-fluid variables.

The Ohm's law we derive here assumes quasineutrality. (One can obtain a more general Ohm's law.) Although Ohm's law does not assume magnetostatics, it is not very useful unless this assumption is made. On the basis of the quasineutrality assumption alone, Ohm's law gives an evolution equation for the current. Such a system is hardly worthy to be called a simplification of the two-fluid equations, since we accomplish a very small reduction in the number of equations (strictly speaking, quasineutrality only allows us to combine the density evolution equations for each species) in exchange for a system that involves many more terms.

With the magnetostatic assumption, $\mathbf{J} = \mu_0^{-1}\nabla \times \mathbf{B}$, not only can we eliminate the current, but Ohm's law gives us a constitutive relation that allows us to eliminate the electric field $\mathbf{E}$ by expressing it in terms of other variables.

Multiplying the momentum evolution equations for each species by its charge to mass ratio gives a pair of evolution equations for the current of each species:

$$\partial_t \begin{bmatrix} \mathbf{J}_i \\ \mathbf{J}_e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \mathbf{J}_i\mathbf{u}_i + \tilde{m}_i^{-1}\mathbb{P}_i \\ \mathbf{J}_e\mathbf{u}_e - \tilde{m}_e^{-1}\mathbb{P}_e \end{bmatrix} = \frac{1}{r} \begin{bmatrix} \tilde{m}_i^{-1}\sigma_i(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) \\ -\tilde{m}_e^{-1}\sigma_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) \end{bmatrix} + \begin{bmatrix} \tilde{m}_i^{-1}\mathbf{R}_i \\ -\tilde{m}_e^{-1}\mathbf{R}_e \end{bmatrix},$$

Adding these equations will give an evolution equation for the net current. We will invoke quasineutrality, $-\sigma_e \approx \sigma_i \approx ne = \rho/(\tilde{m}_i + \tilde{m}_e)$. Define $\mathbf{J}'_s := \sigma_s \mathbf{w}_s$. So $\mathbf{J}_s := \sigma_s\mathbf{u} + \mathbf{J}'_s$ and $\sum_s \mathbf{J}'_s = \mathbf{J}$. Then $\sum_s \mathbf{J}_s\mathbf{u}_s = \sum_s \sigma_s\mathbf{u}_s(\mathbf{u} + \mathbf{w}_s) = \mathbf{J}\mathbf{u} + \sum_s \sigma_s\mathbf{u}_s\mathbf{w}_s = \mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} + \sum_s \sigma_s\mathbf{w}_s\mathbf{w}_s$. So the sum is

$$\begin{aligned} &\partial_t\mathbf{J} + \nabla \cdot \big(\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} + \sigma_i\mathbf{w}_i\mathbf{w}_i + \sigma_e\mathbf{w}_e\mathbf{w}_e + \tilde{m}_i^{-1}\mathbb{P}_i - \tilde{m}_e^{-1}\mathbb{P}_e\big) \\ &= \frac{\rho}{r\tilde{m}_i\tilde{m}_e}(\mathbf{E} + \mathbf{u} \times \mathbf{B} + (\mathbf{w}_i + \mathbf{w}_e) \times \mathbf{B}) + (\tilde{m}_i^{-1} + \tilde{m}_e^{-1})\mathbf{R}_i. \end{aligned}$$

Recalling that $\mathbf{w}_i = \tilde{m}_e\mathbf{J}/\rho$ and $\mathbf{w}_e = \tilde{m}_i\mathbf{J}/\rho$ and recalling the approximate constitutive relation for the drag force $R_i = -\nu n\mathbf{J}/e$, this becomes

$$\begin{aligned} &\partial_t\mathbf{J} + \nabla \cdot \big(\mathbf{u}\mathbf{J} + \mathbf{J}\mathbf{u} + (\tilde{m}_e - \tilde{m}_i)\mathbf{J}\mathbf{J}/\rho + \tilde{m}_i^{-1}\mathbb{P}_i - \tilde{m}_e^{-1}\mathbb{P}_e\big) \\ &= \frac{\rho}{r\tilde{m}_i\tilde{m}_e}(\mathbf{E} + \mathbf{u} \times \mathbf{B} + -(\tilde{m}_i - \tilde{m}_e)\mathbf{J} \times \mathbf{B}/\rho - r\nu\mathbf{J}/e^2). \end{aligned}$$

Solving for the electric field gives:

$$\mathbf{E} = \frac{r\nu}{e^2}\mathbf{J} + \mathbf{B} \times \mathbf{u} + \frac{\tilde{m}_i - \tilde{m}_e}{\rho}\mathbf{J} \times \mathbf{B} + \frac{r}{\rho}\nabla \cdot (\tilde{m}_e\mathbb{P}_i - \tilde{m}_i\mathbb{P}_e) + \frac{r\tilde{m}_i\tilde{m}_e}{\rho}\Big(\partial_t\mathbf{J} + \nabla \cdot \big(\mathbf{u}\mathbf{J} + \mathbf{J}\mathbf{u} + \frac{\tilde{m}_e - \tilde{m}_i}{\rho}\mathbf{J}\mathbf{J}\big)\Big).$$

5 Conservation form

We assume $\sigma \approx 0$ for simplicity in the following development.

To put energy balance in conservation form, we use Ampere's law to eliminate the current and rewrite minus the source term as the time derivative of something we call magnetic energy plus the divergence of something called the Poynting vector, which we regard as the flux of electromagnetic energy.

$$
\begin{aligned}
-(\mathbf{J}/r)\cdot\mathbf{E} &= -\mu_0^{-1}(\nabla\times\mathbf{B})\cdot\mathbf{E} \\
&= -\mu_0^{-1}(-\nabla\cdot(\mathbf{E}\times\mathbf{B})+\mathbf{B}\cdot\nabla\times\mathbf{E}) \\
&= -\mu_0^{-1}(-\nabla\cdot(\mathbf{E}\times\mathbf{B})-\mathbf{B}\cdot\partial_t\mathbf{B}) \\
&= \mu_0^{-1}[\nabla\cdot(\mathbf{E}\times\mathbf{B})+\partial_t B^2/2].
\end{aligned}
$$

To put momentum balance in conservation form, we use Ampere's law to eliminate the current from minus the momentum source term and rewrite it as the divergence of something we call the magnetic pressure tensor (i.e. flux of magnetic momentum):

$$
\begin{aligned}
-(\mathbf{J}/r)\times\mathbf{B} &= -\mu_0^{-1}(\nabla\times\mathbf{B})\times\mathbf{B} \\
&= -\mu_0^{-1}(\mathbf{B}\cdot\nabla\mathbf{B}-(\nabla\mathbf{B})\cdot\mathbf{B}) \\
&= \mu_0^{-1}\nabla\cdot(\mathbb{I}B^2/2-\mathbf{B}\mathbf{B}).
\end{aligned}
$$

We can now write the MHD equations in conservation form:

$$
\begin{aligned}
&\partial_t\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \tilde{\mathcal{E}}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+\tilde{m}_e\tilde{m}_i\mathbf{J}\mathbf{J}/\rho+\mathbb{P}+\mu_0^{-1}(\mathbb{I}B^2/2-\mathbf{B}\mathbf{B})\\ \mathbf{u}\mathcal{E}+\mathbf{u}\cdot\mathbb{P}+\sum_s(\mathbf{w}_s\mathcal{E}_s+\mathbf{w}_s\cdot\mathbb{P}_s)+\mathbf{q}+\mu_0^{-1}\mathbf{E}\times\mathbf{B}\end{bmatrix}=0 \quad \text{and} \\
&\partial_t\mathbf{B}+\nabla\times\mathbf{E}=0,
\end{aligned}
$$

where $\tilde{\mathcal{E}} := \mathcal{E}+\mu_0^{-1}\mathbf{B}^2/2$ is the total energy, and we recall that $\mu_0^{-1}=c^2\lambda^2 r$.

6 MHD equations: hyperbolic flux form

For numerical shock-capturing purposes, we decompose the flux as the sum of a function of non-differentiated state variables (which we will call the *hyperbolic* flux, since it turns out to have real eigenvalues and a full set of eigenvectors) and a function of differentiated state variables. For this purpose we write Ohm's law as $\mathbf{E}=\mathbf{B}\times\mathbf{u}+\mathbf{E}'$, where $\mathbf{E}'$ denotes Ohm's law in the reference frame of the fluid. Substituting Ohm's law into Faraday's law (i.e., taking the curl of Ohm's law) identifies the hyperbolic flux of magnetic field: $\nabla\times(\mathbf{B}\times\mathbf{u})=\nabla\cdot(\mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u})$. So:

$$
\begin{aligned}
\partial_t\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \tilde{\mathcal{E}}\\ \mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+\mathbb{I}\tilde{p}_{\mathrm{MHD}}-\mu_0^{-1}(\mathbf{B}\mathbf{B})\\ \mathbf{u}(\tilde{\mathcal{E}}+\tilde{p}_{\mathrm{MHD}})-\mu_0^{-1}\mathbf{B}\mathbf{B}\cdot\mathbf{u}\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\end{bmatrix} &= \nabla\cdot\begin{bmatrix}0\\ (\tilde{m}_e\tilde{m}_i/\rho)(\mathbb{I}J^2/3-\mathbf{J}\mathbf{J})\\ \mathbf{u}\tilde{m}_e\tilde{m}_iJ^2/(3\rho)-\mathbf{q}^{\mathrm{d}}_{\mathrm{disp}}-\mu_0^{-1}\mathbf{E}_{\mathrm{disp}}\times\mathbf{B}\\ \underline{\underline{\epsilon}}\cdot\mathbf{E}_{\mathrm{disp}}\end{bmatrix} \\
&\quad+\nabla\cdot\begin{bmatrix}0\\ \mathbb{T}\\ \mathbf{u}\cdot\mathbb{T}-\mathbf{q}-\mathbf{q}^{\mathrm{d}}_{\mathrm{diff}}-\mu_0^{-1}\mathbf{E}_{\mathrm{diff}}\times\mathbf{B}\\ \underline{\underline{\epsilon}}\cdot\mathbf{E}_{\mathrm{diff}}\end{bmatrix},
\end{aligned}
$$

where $\mathbb{T}$ is the viscous stress tensor (which is typically assumed to depend linearly and isotropically on the symmetric part of the velocity gradient), and $\mathbf{q}=-K\nabla T$ is the heat flux. Here $\tilde{\mathcal{E}}=$

$\mathcal{E} + \mu_0^{-1}B^2/2$, $\mathcal{E} = (3/2)p + (1/2)\rho u^2$, $\tilde{p}_{\mathrm{MHD}} = p + \mu_0^{-1}B^2/2$ is the total pressure, $\mathbf{J} = \mu_0^{-1}\nabla \times \mathbf{B}$, $\mathbf{w}_i = \tilde{m}_e\mathbf{J}/\rho$, $\mathbf{w}_e = -\tilde{m}_i\mathbf{J}/\rho$, and $\mathbf{E}' = \mathbf{E}_{\mathrm{diff}} + \mathbf{E}_{\mathrm{disp}}$, where

$$\begin{aligned}
\mathbf{E}_{\mathrm{diff}} &= r\nu\mathbf{J} - \frac{r}{\rho}\nabla\cdot(\tilde{m}_e\mathbb{T}_i - \tilde{m}_i\mathbb{T}_e),\\
\mathbf{E}_{\mathrm{disp}} &= \frac{\tilde{m}_i - \tilde{m}_e}{\rho}\mathbf{J}\times\mathbf{B} + \frac{r}{\rho}\nabla(\tilde{m}_e p_i - \tilde{m}_i p_e) + \frac{r\tilde{m}_i\tilde{m}_e}{\rho}\Big(\partial_t\mathbf{J} + \nabla\cdot\big(\mathbf{uJ} + \mathbf{Ju} + \frac{\tilde{m}_e - \tilde{m}_i}{\rho}\mathbf{JJ}\big)\Big).
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{q}^{\mathrm{d}}_{\mathrm{diff}} &= -\frac{\mathbf{J}}{\rho}\cdot(\tilde{m}_e\mathbb{T}_i - \tilde{m}_i\mathbb{T}_e),\\
\mathbf{q}^{\mathrm{d}}_{\mathrm{disp}} &= \tilde{m}_i\tilde{m}_e\frac{\mathbf{JJ}}{\rho}\cdot(\mathbf{u} + \frac{\tilde{m}_e - \tilde{m}_i}{2}\frac{\mathbf{J}}{\rho}) + \frac{5}{2}\frac{\mathbf{J}}{\rho}(\tilde{m}_e p_i - \tilde{m}_i p_e)
\end{aligned}$$

I'm not confident in how I've separated out the differentiated source terms into diffusive terms and dispersive terms. I'm making the assumption (true?) that if we start our derivation from a collisionless ideal two-fluid model all terms in the resulting one-fluid model will be dispersive, and that all terms in our model that come from diffusive terms in the two-fluid model are diffusive.[1]

Observe that the only place where the gyroradius r appears in the MHD equations is implicitly in the $\mathbf{E}'$ terms of Ohm's law. It looks like a small gyroradius gives us $\mathbf{E}' = r\nu\mathbf{J} + \frac{\tilde{m}_i - \tilde{m}_e}{\rho}\mathbf{J}\times\mathbf{B}$, giving Hall MHD. A large ρ seems to eliminate the remaining dispersive terms, including the $\mathbf{J}\times\mathbf{B}$ Hall term, and implies that the $\mathbf{w}_s$ can be neglected. The collisionless assumption eliminates the diffusive terms such as the resistivity term $r\nu\mathbf{J}$ and should eliminate the viscous stress tensor.

Shocks can develop when the hyperbolic flux dominates the nonhyperbolic flux; putting our equations in hyperbolic conservative flux form helps us write numerical methods that resolve computed shocks sharply and propagate them at the correct speeds.

References

[Woods] L. C. Woods, Physics of Plasmas, Wiley-VCH Verlag, 2004.

[1] To determine whether this assumption is true it would apparently be necessary to close this system by specifying (or neglecting) the pressure term. I would expect the diffusive terms to produce entropy while I would expect the dispersive terms not to produce entropy, based on the fact that they arise from terms in the Boltzmann (or two-fluid) model which do or do not generate entropy accordingly. Fluid entropy can be defined by assuming the form of the velocity distribution (Maxwellian) and then integrating the entropy over velocity space. The Vlasov equation conserves entropy; this leads to the fact that moments of the Vlasov equation also conserve entropy. So it would seem that averages of these moments (the one-fluid model) should also conserve entropy. Scalar pressure assumes no viscosity. Fluid-dynamic entropy evolution is usually derived from thermal energy (i.e. pressure) evolution, which is obtained by subtracting kinetic energy evolution from energy evolution. I need to do this for extended MHD as I have done for two-fluid gas dynamics e.g. in my note on the ten moment closure.