Steady reconnection problems

by E. Alec Johnson, Feb 2011

1 PHS problem

Pei, Horiuchi, and Sato in [8, 7] introduced the following driven reconnection problem which asymptotes to steady reconnection. I refer to this problem by their initials, PHS. Ishizawa et al. subsequently studied this problem in [5] and [4].

1.1 Plasma parameters

Reference [8] uses a mass ratio of $m_\mathrm{i}/m_\mathrm{e} = 25$. Reference [5] also uses $m_\mathrm{i}/m_\mathrm{e} = 200$.

1.2 Nondimensionalization

In the references time is is nondimensionalized by ion cyclotron frequency $\omega_{ci} := \frac{eB_0}{m_\mathrm{i}}$ (e.g.[7], [5]) or electron cyclotron frequency $\omega_{ce} := \frac{eB_0}{m_\mathrm{e}}$ (e.g.[8]). These references nondimensionalize space using the ion gyroradius.

The GEM problem uses ion skin depth δ_i rather than ion gyroradius to nondimensionalize space. That is, it uses ion Alfvén speed rather than ion thermal velocity to nondimensionalize velocity. For a Harris sheet equilibrium these are equivalent, because the thermal pressure at the neutral line balances the magnetic pressure far from the neutral line. To be specific: The ion skin depth is the gyroradius of an ion moving at the Alfvén speed, whereas the gyroradius is the gyroradius of an ion moving at the thermal speed. The Alfvén speed is the square root of twice the *magnetic* pressure divided by the density, whereas the thermal speed is the square root of (once or twice) the *thermal* pressure divided by the density.

The skin depth may be preferable to gyroradius as more well-defined. It does not depend on the magnetic field (or temperature) and is completely determined by particle density: $\delta_\mathrm{i}^2 := \frac{m_\mathrm{i}}{\mu_0 n_0 e^2}$.

1.3 Domain

The domain is the rectangle $[-x_b, x_b] \times [-y_b, y_b]$ in the x-y plane. Reference [8] chooses $x_b/y_b = 6$. To avoid the formation of islands [5] adopts a narrower x_b/y_b of 2.

1.4 Initial conditions

The initial condition is a one-dimensional Harris sheet equilibrium (constant total pressure with unidirectional magnetic field constant along the direction it points):

$$B_x(y) = B_0 \tanh(y/y_h),$$
$$p(y) = p_B \operatorname{sech}^2(y/y_h) + p_\infty,$$

where $p_B := \frac{B_0^2}{2\mu_0}$, the pressure variation, is also the magnetic pressure; the background pressure p_∞ is zero, y_h is the scale height, B_0 is magnetic field, and μ_0 is magnetic permeability. The temperature is uniform and equal in both species, $T_0 := T_{\mathrm{i}0} = T_{\mathrm{e}0}$. (Since $p_\mathrm{i} = n_\mathrm{i} T_\mathrm{i}$ and $p_\mathrm{e} = n_\mathrm{e} T_\mathrm{e}$ this implies that number density is proportional to pressure.)

Reference [8] sets $\omega_{pe}/\omega_{\mathrm{ce}}$ (i.e. $c/V_{A\mathrm{e}} = (c/V_A)\sqrt{m_\mathrm{e}/m_0}$) equal to 3.5 and sets "$y_h = 0.8 y_b \approx 3 r_{c\mathrm{i}}$", where $r_{c\mathrm{i}}$ is the ion cyclotron radius. Recall that $r_{c\mathrm{i}} := v_{t\mathrm{i}}/\omega_{c\mathrm{i}}$, where $v_{t\mathrm{i}} := \sqrt{T_0/m_\mathrm{i}}$ is the ion thermal velocity. The definition of the thermal velocity seems to vary in the literature — some use $v_{t\mathrm{i}}^2 := 2T_0/m_0$ (see e.g. [1, 9]) and some use $v_{t\mathrm{i}}^2 := T_0/m_0$ (see e.g. [3]). The ion cyclotron frequency is $\omega_{c\mathrm{i}} = \frac{e\mathbf{B}_0}{m_\mathrm{i}}$.

1.5 Boundary conditions

1.5.1 Upstream boundary

Electromagnetic field. Define $E_{zd}(t,x) := E_z(t,x,y_b)$ and $B_{yd}(t,x) := B_y(t,x,y_b)$. At the upstream boundary the external "drive" electric field $E_{zd}(t,x)$ is applied in the z direction at $y = \pm y_b$. The electric field parallel to the outflow axis is zero ($E_x = 0$), and the component parallel to the inflow axis is constrained by the divergence constraint which implies that $\partial_y E_y = 0$.

The B_y component is determined from E_z on the inflow boundary via Faraday's law. Specifically, the y component of Faraday's law says that $\partial_t B_y = \partial_x E_z$. Integrating from $(0, y_b)$ to (x, y_b) along the line $(y = y_b)$ gives

$$\partial_t \int_0^x B_{yd} = [E_{zd}]_{x=0}^x. \tag{1}$$

Horiuchi driving electric field. The following description of the driving electric field is obtained from Professor Horiuchi [Ritoku Horiuchi, private communication, March 2011] in reference to [8]. The driving electric field imposed at the input boundary consists of uniform and bell-shaped components as

$$E_{zd}(t,x) = f_\infty(t) + f_\Delta(t)\frac{\cos\phi(x)+1}{2}, \tag{2}$$

where

$$\phi(x) := \begin{cases} -\pi & \text{if } x \le -x_d, \\ \pi x/x_d & \text{if } -x_d \le x \le x_d, \\ \pi & \text{if } x \ge x_d. \end{cases}$$

The amplitudes $f_\infty(t)$ and $f_\Delta(t)$ grow with time in proportion to

$$\frac{1-\cos(\frac{\pi t}{t_1})}{2} \text{ for } 0 \le t \le t_1.$$

That is, for $0 \le t \le t_1$

$$f_\infty(t) = f_\infty(t_1)\frac{1-\cos\theta(t)}{2} \text{ and}$$
$$f_\Delta(t) = f_\Delta(t_1)\frac{1-\cos\theta(t)}{2}, \text{ where}$$
$$\theta(t) := \frac{\pi t}{t_1}.$$

When $t = t_1$ the driving field reaches E_0, i.e.

$$E_{zd}(t_1, 0) = E_0 = f_\infty(t_1) + f_\Delta(t_1).$$

Here $f_\infty(t_1) = 0.3E_0$ in [8].

For $t_1 \le t \le t_2$, $f_\infty(t)$ increases toward $E_0(0)$ and $f_\Delta(t)$ decreases toward zero while keeping the condition $E_{zd}(t,0) = E_0$. Specifically, for $t_1 \le t \le t_2$

$$f_\infty(t) = f_\infty(t_1) + (E_0 - f_\infty(t_1))\left(\frac{1-\cos\psi(t)}{2}\right),$$
$$f_\Delta(t) = f_\Delta(t_1) - f_\Delta(t_1)\left(\frac{1-\cos\psi(t)}{2}\right)$$
$$= f_\Delta(t_1)\left(\frac{1+\cos\psi(t)}{2}\right), \text{ where}$$
$$\psi(t) = \pi\frac{t-t_1}{t_2-t_1}.$$

In [8], $t_1 = 100/3.5 \approx 28.6$ and $t_2 = 500/3.5 \approx 143$.

For $t \ge t_2$ the field $E_{zd}(t,x)$ becomes constant in time and space.

Inferred magnetic field. From equations (1) and (2) we can infer B_y on $y = y_b$:

$$[E_{zd}]_{x=0}^x = f_\Delta(t)\frac{\cos\phi(x)-1}{2}.$$

So

$$\int_0^x B_{yd} = \left(\int_0^t f_\Delta\right)\frac{\cos\phi(x)-1}{2}.$$

But

$$\begin{aligned} \int_0^t f_\Delta &= f_\Delta(t_1)\int_0^t \left(\frac{1-\cos\theta(t)}{2}\right) && \text{for } 0 \le t \le t_1 \\ &= \frac{f_\Delta(t_1)}{\theta'}\left(\frac{\theta(t)-\sin\theta(t)}{2}\right) && \text{for } 0 \le t \le t_1 \\ &= t_1 f_\Delta(t_1)/2 && \text{for } t = t_1. \end{aligned}$$

and

$$\begin{aligned} \int_{t_1}^t f_\Delta &= f_\Delta(t_1)\int_{t_1}^t \left(\frac{1+\cos\psi(t)}{2}\right) && \text{for } t_1 \le t \le t_2 \\ &= \frac{f_\Delta(t_1)}{\psi'}\left(\frac{\psi(t)+\sin\psi(t)}{2}\right) && \text{for } t_1 \le t \le t_2 \\ &= (t_2-t_1) f_\Delta(t_1)/2 && \text{for } t = t_2; \end{aligned}$$

that is,

$$\int_0^t f_\Delta = \begin{cases} \frac{f_\Delta(t_1)}{\theta'}\left(\frac{\theta(t)-\sin\theta(t)}{2}\right) & \text{for } 0 \le t \le t_1, \\ \frac{t_1 f_\Delta(t_1)}{2} + \frac{f_\Delta(t_1)}{\psi'}\left(\frac{\psi(t)+\sin\psi(t)}{2}\right) & \text{for } t_1 \le t \le t_2. \\ t_2 f_\Delta(t_1)/2 & \text{for } t \ge t_2, \end{cases}$$

which shows that in steady state the magnetic flux exiting the top is proportional to the strength and duration of the perturbation of the driving electric field. Differentiating,

$$B_{yd} = \underbrace{\frac{-\phi' \int_0^t f_\Delta}{2}}_{\text{Call } \beta(t)} \sin\phi(x)$$

The "early-phase nonuniformity scale" (driving region width) x_d may vary. Reference [8] obtains steady reconnection when x_d/x_b is 0.42 and 0.62. When $x_d/x_b = 0.83$ their reconnection is intermittent due to the formation and ejection of magnetic islands.

Presumably B_z is zero on the inflow boundary (although this does not seem to be stated). For B_x it is assumed that on the boundary the z component of current plus displacement current can be neglected, so $\partial_y B_x \approx \partial_x B_y$ on $y = y_b$.

Reference [5] says, "The external field $E_{zd}(x,t)$ is programmed to evolve from zero to a constant value during an early phase $0 \leq t \leq \tau_A$, where $\tau_A = y_b/V_A$ is Alfvén transit time and V_A is an initial Alfvén velocity."

Velocity. The inflow velocity is the $\mathbf{E} \times \mathbf{B}$ drift velocity:

$$\mathbf{u} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \text{ at } y = \pm y_b,$$

which is the component perpendicular to the magnetic field of the velocity of a frozen-in flux; so we assume that the component parallel to the magnetic field is zero.

Density and temperature. Though not explicitly stated, I would suppose that the inflow density and temperature are those of the initial condition.

Summary of inflow boundary conditions:

$$\begin{aligned}
\rho_s &= m_s p(y_b)/T_0, \\
\mathbf{u}_s &= \mathbf{E} \times \mathbf{B}/B^2, \\
p_s &= p(y_b)/2?, \\
\partial_y B_x &= \partial_x B_y = \beta(t)\cos(\phi(x))\phi'(x), \\
B_y &= \beta(t)\sin(\phi(x)), \\
B_z &= 0(?), \\
E_x &= 0, \\
\partial_y E_y &= 0, \\
E_z &= f_\infty(t) + f_\Delta(t)\frac{\cos\phi(x)+1}{2}, \\
\partial_y \psi &= 0,
\end{aligned}$$

where we note that

$$\phi'(x) = \begin{cases} \pi/x_d & \text{if } -x_d \leq x \leq x_d, \\ 0 & \text{otherwise.} \end{cases}$$

1.5.2 Downstream boundary

Reference [5] states, "The field quantities E_x, E_y, and $\partial_x E_z$ are continuous at the downstream boundary. These conditions enable magnetic islands to go through the boundary. The remaining components of the field quantities are given by solving the Maxwell equations at the boundary." (But [6] states, "At the downstream boundary, the field quantities E_x, $\partial_x E_y$ and $\partial_x E_z$ are continuous".) This does not seem to be a full specification of outflow boundary conditions.

One-dimensional outflow boundary conditions simply copy from the neighboring cell. (The reason this works is that copying works for inflowing characteristics, and for outflowing characteristics it makes no difference what you do and so copying again works).

In multiple dimensions the characteristics depend on the choice of direction vector. Unless the solution is constant perpendicular to such a direction vector copy boundary conditions don't seem justified as a way to implement open boundaries. The general attitude in the computational community seems to be that if you have significant transverse derivatives at an artificial boundary you need to go back and redesign the problem.

Copy boundary conditions are basically equivalent to setting partials with respect to x equal to zero. The induction equation says:

$$\begin{aligned}
&\partial_t B_x + \partial_y E_z = 0, \\
&\partial_t B_y - \partial_x E_z = 0, \\
&\partial_t B_z + \partial_x E_y - \partial_y E_x = 0.
\end{aligned}$$

Setting $\partial_x E_z$ equal to zero would not allow B_y to change, which perhaps explains the quote above.

To allow magnetic islands to exit the domain, can we populate ghost cells by extrapolating rather than copying from neighbors? Is this stable?

Summary of outflow boundary conditions:

$$\begin{aligned}
\rho_s &= \text{copy?} \\
\mathbf{u}_s &= \text{copy?} \\
p_s &= \text{copy?} \\
B_x &= \text{copy?} \\
B_y &= \text{copy?} \\
B_z &= \text{copy?} \\
E_x &= \text{copy?} \\
E_y &= \text{copy or extrapolate?} \\
E_z &= \text{extrapolate} \\
\psi &= \text{copy?}
\end{aligned}$$

The most sophisticated relevant treatment of outflow boundary conditions for PIC that I can find is in [1]. For Hall MHD [2] sets normal derivatives of all quantities to zero (i.e. uses copy boundaries, I assume) for both inflow and outflow to study steady reconnection.

2 Huba-Rudakov problem

I rename variables here to be more like the variable names typically used for Harris sheet perturbations on a square domain. For some reason [2] swaps the x and y axes from the more standard choice of axis names that I use here.

2.1 Initial condition

Initial conditions.

Equilibrium. Temperature $T = T_0$ is uniform. Pressure and magnetic pressure are in balance.

$$\begin{aligned}
&B_x(y) = B_0 \tanh(y/L),\\
&p(y) = p_B \operatorname{sech}^2(y/L) + p_\infty, \text{ where } p_B := \frac{B_0^2}{2\mu_0},\\
&p_0 := p(0)\\
&p_0 = p_B + p_\infty.
\end{aligned}$$

The problem specifies that

$$p_\infty = 0.2p_0, \text{ so } p_B = 0.8p_0.$$

Time is normaled to the ion gyrofrequency $\omega_{ci} := \frac{eB_0}{m_i}$, velocity is normaled to the ion Alfvén speed v_{Ai}, and thus length is normaled to the ion inertial length v_{Ai}/ω_{ci}. Using these units the size of the simulation box is given by

$$L_x = 84 \text{ and } L_y = 70.$$

The magnetic field is perturbed by

$$\begin{aligned}
&\delta\mathbf{B} = \nabla\times(\hat{z}\psi) = -\hat{z}\times\nabla\psi, \text{ where}\\
&\psi = \psi_0 \cos\left(\frac{2\pi x}{L_x}\right)\cos\left(\frac{\pi y}{L_y}\right), \text{ where}\\
&\psi_0 := \delta B\frac{L_x}{2\pi}, \text{ where } \delta B := 0.1B_0.
\end{aligned}$$

Thus the perturbations of the components of the magnetic field are

$$\begin{aligned}
&\delta B_x = \frac{-\delta B}{2}\frac{L_x}{L_y}\cos\left(\frac{2\pi x}{L_x}\right)\sin\left(\frac{\pi y}{L_y}\right),\\
&\delta B_y = \delta B \sin\left(\frac{2\pi x}{L_x}\right)\cos\left(\frac{\pi y}{L_y}\right).
\end{aligned}$$

(There is a missing minus sign in the definition of the perturbation in [2].)

They indicate the ion thermal speed as a fraction of the Alfvén speed:

$$C_i = 0.41 v_{Ai}$$

where $C_i := (2T/m_i)^{1/2}$ is the ion thermal speed. The constant 0.41 is perhaps a typo. For a Harris sheet the magnetic pressure and the thermal pressure balance one another. So there is a simple relationship between the thermal speed and the Alfvén speed. So one doesn't really have freedom to set the thermal speed. Recall that $v_{Ai}^2 = \frac{2p_B}{\rho_i}$, where $p_B = \frac{B_0^2}{2\mu_0}$ is the magnetic pressure, and note that the thermal speed is $C_i^2 = \frac{2p_i}{\rho_i}$. So the relationship $C_i = .41 v_{Ai}$ implies that $p_i = .41^2 p_B = .168 p_B$, which does not seem plausible given that $p_i + p_e$ should roughly balance p_B.

They state, "Zero-gradient boundary conditions are used for all variables in both the x and y directions ($\partial/\partial_x = 0$ and $\partial/\partial_y = 0$)". I assume that they mean that zero normal-derivative boundary conditions are used at each boundary. They then say, "Physically, this implies that there is no acceleration across the boundaries."

I cannot find where they specify the mass ratio m_i/m_e. Probably they assume the limit as m_e/m_i goes to zero, which is typical for a study using Hall MHD (which is what they are doing).

References

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