Source term for two-fluid plasma

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I have computationally verified the solutions in part B of this document with a third-order Runge-Kutta ODE solver.

Part A: modeling

1 Basic Equations

Neglecting spatial derivatives, the two-fluid equations are the source terms of Maxwell's equations coupled to the source terms of the gas-dynamics equations for each species.

Maxwell's source term equations assert that the magnetic field is constant, the displacement current balances the net electrical current, and the electric correction potential ramps linearly in response to net charge (at an extremely stiff rate):

$$
\begin{aligned}
&\partial_t \mathbf{B} = 0, \\
&\partial_t \mathbf{E} = -\mathbf{J}/\varepsilon_0, \\
&\partial_t \phi = \sigma(\chi c)^2/\varepsilon_0 - \varepsilon_2 \phi,
\end{aligned}
$$

where $\mathbf{B}$ is magnetic field, $\mathbf{E}$ is electric field, $\mathbf{J} = \mathbf{J}_\mathrm{i} + \mathbf{J}_\mathrm{e}$ is net current, $\sigma = \sigma_\mathrm{i} + \sigma_\mathrm{e}$ is net charge, $\sigma_\mathrm{i} = en_\mathrm{i}$ is ion charge, $\sigma_\mathrm{e} = -en_\mathrm{e}$ is electron charge, e is the charge on a proton, n_i is ion particle density, n_e is electron particle density, $\mathbf{J}_\mathrm{i} = \sigma_\mathrm{i}\mathbf{u}_\mathrm{i}$ is ion current, $\mathbf{J}_\mathrm{e} = \sigma_\mathrm{e}\mathbf{u}_\mathrm{e}$ is ion current, $\mathbf{u}_\mathrm{i}$ is ion bulk velocity, $\mathbf{u}_\mathrm{e}$ is electron bulk velocity, ϕ is electric divergence correction potential, $c\chi \geq c$ is the correction potential speed, and ε_2 is the maximal decay rate of the electric correction potential.

The source terms for density assert that the densities (whether mass density or particle density or charge density) remain constant:

$$
\begin{aligned}
&\partial_t \rho_\mathrm{s} = 0, \text{ i.e.,} \\
&\partial_t n_\mathrm{s} = 0, \text{ i.e.,} \\
&\partial_t \sigma_\mathrm{s} = 0.
\end{aligned}
$$

The momentum equation for generic species s is

$$\partial_t(\rho_\mathrm{s}\mathbf{u}_\mathrm{s}) = \frac{q_\mathrm{s}}{m_\mathrm{s}}\rho_\mathrm{s}(\mathbf{E} + \mathbf{u}_\mathrm{s} \times \mathbf{B}) + \overrightarrow{\mathbf{R}}_\mathrm{s},$$

where s is the species of interest, $\frac{q_\mathrm{s}}{m_\mathrm{s}}$ denotes charge to mass ratio, and $\overrightarrow{\mathbf{R}}_\mathrm{s}$ denotes the drag force on species s due to collisions with other species.

Under the approximating (or time-splitting) assumption that the evolution of the drag coefficient does not depend on the pressure evolution equations, the momentum equations in conjunction with the electric field equation can be solved independently.

Five-moment gas has an evolution equation for scalar pressure,

$$\frac{3}{2}\partial_t p_\mathrm{s} = Q_\mathrm{s}^\mathrm{f} + Q_\mathrm{s}^\mathrm{t}, \text{ i.e., } \partial_t T_\mathrm{s} = \frac{2}{3n_\mathrm{s}}(Q_\mathrm{s}^\mathrm{f} + Q_\mathrm{s}^\mathrm{t})$$

where we have used that $p_\mathrm{s} = n_\mathrm{s} T_\mathrm{s}$ and where Q_s^f denotes frictional heating due to the drag force and Q_s^t denotes heating or cooling due to thermal equilibration among species.

Ten-moment gas dynamics instead has an evolution equation for tensor pressure:

$$
\begin{aligned}
&\partial_t \mathbb{P}_\mathrm{s} = (q_\mathrm{s}/m_\mathrm{s})2\mathrm{Sym}(\mathbb{P}_\mathrm{s} \times \mathbf{B}) + \mathbb{R}_\mathrm{s} + \mathbb{Q}_\mathrm{s}^\mathrm{f} + \mathbb{Q}_\mathrm{s}^\mathrm{t}, \quad \text{i.e.,} \\
&\partial_t \mathbb{T}_\mathrm{s} = (q_\mathrm{s}/m_\mathrm{s})2\mathrm{Sym}(\mathbb{T}_\mathrm{s} \times \mathbf{B}) + (\mathbb{R}_\mathrm{s} + \mathbb{Q}_\mathrm{s}^\mathrm{f} + \mathbb{Q}_\mathrm{s}^\mathrm{t})/n_\mathrm{s},
\end{aligned}
$$

where Sym denotes symmetric part of its argument tensor, $\mathbb{R}_\mathrm{s}$ denotes relaxation toward isotropy due to intraspecies collisions, $\mathbb{Q}_\mathrm{s}^\mathrm{f}$ is generalized frictional heating, and $\mathbb{Q}^\mathrm{t}$ is generalized thermal equilibration.

2 Closure

Collisional closure coefficients of linear closure relations are most naturally related to the relaxation periods they effect. In each case the relaxation period is of the form

$$\tau = \tilde{\tau}_0 \frac{(\mathrm{mass})^2}{n}(\mathrm{vel})^3,$$

where (mass) is an appropriate mass (or average of masses on a geometric scale), (vel) is an appropriate average velocity, and $\tilde{\tau}_0$ is a proportionality constant on the order of

$$\tau_0 := 3(2\pi)^{3/2}\frac{\varepsilon_0^2}{e^4}\frac{1}{\ln\Lambda};$$

that is, $\tilde{\tau}_0 = \alpha_0\tau_0$, with α_0 a different number on the order of order 1 for each relaxation period formula.

2.1 Resistivity

Resistive drag effects equilibration of velocities and effects resistivity.

Assume that the resistive drag force is proportional to the density of particles and the interspecies drift velocity: $\overrightarrow{\mathbf{R}}_{\text{ie}} = \eta e^2 n_\text{i} n_\text{e}(\mathbf{u}_\text{e} - \mathbf{u}_\text{i})$. The coefficient η is called the resistivity. To see that it is the resistivity, look at Ohm's law (equivalently current balance) when resistive drag balances electric field:

$$\begin{aligned}
0 &= en_\text{i}\mathbf{E} + \overrightarrow{\mathbf{R}}_{\text{ie}}, \\
0 &= -en_\text{e}\mathbf{E} - \overrightarrow{\mathbf{R}}_{\text{ie}}.
\end{aligned}$$

So charge neutrality $n_\text{i} = n_\text{e} =: n$ holds and

$$\mathbf{E} = -\frac{\overrightarrow{\mathbf{R}}_{\text{ie}}}{en} = \eta en(\mathbf{u}_\text{i} - \mathbf{u}_\text{e}) = \eta \mathbf{J},$$

where $\mathbf{J}$ is net current.

To see how the resistivity is related to the velocity equilibration time, consider momentum balance when the drag force alone is present:

$$\begin{aligned}
&m_\text{e} n_\text{e} \partial_t \mathbf{u}_\text{e} = \eta e^2 n_\text{i} n_\text{e}(\mathbf{u}_\text{i} - \mathbf{u}_e) \quad \text{and} \\
&m_\text{i} n_\text{i} \partial_t \mathbf{u}_\text{i} = \eta e^2 n_\text{i} n_\text{e}(\mathbf{u}_\text{e} - \mathbf{u}_i).
\end{aligned}$$

That is,

$$\begin{aligned}
&\partial_t \mathbf{u}_\text{e} = \eta e^2 n_\text{i}(\mathbf{u}_\text{i} - \mathbf{u}_e)/m_\text{e} \quad \text{and} \\
&\partial_t \mathbf{u}_\text{i} = \eta e^2 n_\text{e}(\mathbf{u}_\text{e} - \mathbf{u}_i)/m_\text{i}
\end{aligned}$$

Again assuming charge neutrality, taking the difference yields

$$\begin{aligned}
&\partial_t(\mathbf{u}_\text{i} - \mathbf{u}_\text{e}) = -\eta e^2 n(\mathbf{u}_\text{i} - \mathbf{u}_e)/\mu, \quad \text{i.e.,} \\
&\partial_t(\mathbf{u}_\text{i} - \mathbf{u}_\text{e}) = -(\mathbf{u}_\text{i} - \mathbf{u}_e)/\tau_\text{slow}
\end{aligned}$$

where $\mu := (m_\text{i}^{-1} + m_\text{e}^{-1})^{-1}$ is reduced mass and where the resistivity evidently is related to the slowing down time by

$$\eta = \frac{\mu}{e^2 n \tau_\text{slow}}.$$

Braginskii gives

$$\tau_\text{slow} = \tilde{\tau}_0 \frac{\sqrt{m_\text{e}}}{n} T_e^{3/2},$$

assuming $m_\text{e} \ll m_\text{i}$. A generalization to include pair plasmas (consistent with the expression below for temperature equilibration period below) is

$$\tau_\text{slow} = \tilde{\tau}_0 \frac{\mu^2}{n} \left(\frac{T_\text{i}}{m_\text{i}} + \frac{T_\text{e}}{m_\text{e}} \right)^{3/2}$$

(note that $\sqrt{\frac{T_\text{i}}{m_\text{i}} + \frac{T_\text{e}}{m_\text{e}}}$ computes an overall root-mean-square velocity), which simplifies to

$$\tau_\text{slow} = \tilde{\tau}_0 \frac{\sqrt{\mu}}{n} T^{3/2}$$

if $T_i = T_e =: T$.

2.2 Energy equilibration

2.2.1 Frictional heating

A closure for the distribution of frictional heating is distribution in inverse proportion to mass:

$$\begin{aligned}
Q^\text{f}_\text{total} &:= Q^\text{f}_\text{i} + Q^\text{f}_\text{e} = \overrightarrow{\mathbf{R}}_\text{i} \cdot (\mathbf{u}_\text{e} - \mathbf{u}_\text{i}), \\
Q^\text{f}_\text{i} &= Q^\text{f}_\text{total} \frac{m_\text{e}}{m_\text{e} + m_\text{i}}, \\
Q^\text{f}_\text{e} &= Q^\text{f}_\text{total} \frac{m_\text{i}}{m_\text{e} + m_\text{i}}.
\end{aligned}$$

2.2.2 Temperature equilibration

Braginskii says that the thermal equilibration period is

$$\begin{aligned}
\tau_\text{temp} &= \tilde{\tau}_0 \frac{m_\text{i} m_\text{e}}{2n} \left(\frac{T_\text{i}}{m_\text{i}} + \frac{T_\text{e}}{m_\text{e}} \right)^{3/2} \\
&= \tau_\text{slow} \frac{m}{2\mu},
\end{aligned}$$

where $m := m_\text{i} + m_\text{e}$ is total particle mass.

A natural linear closure for heating due to thermal equilibration is

$$Q^\text{t}_\text{i} = (3/2) K n_\text{i} n_\text{e} (T_\text{e} - T_\text{i}),$$

where K is a thermal equilibration coefficient. To relate K to τ_temp consider temperature evolution when temperature equilibration alone is present:

$$\begin{aligned}
&\partial_t T_\text{i} = \frac{2}{3n_\text{i}} Q^\text{t}_\text{s}, \quad \text{i.e.,} \\
&\partial_t T_\text{i} = K n_\text{e}(T_\text{e} - T_\text{i}), \quad \text{and} \\
&\partial_t T_\text{e} = K n_\text{i}(T_\text{e} - T_\text{i}).
\end{aligned}$$

Assuming neutrality ($n_\mathrm{i} = n_\mathrm{e} = n$),

$$\partial_t (T_\mathrm{i} - T_\mathrm{e}) = -nK(T_\mathrm{i} - T_\mathrm{e}) \quad \text{i.e.,}$$
$$\partial_t (T_\mathrm{i} - T_\mathrm{e}) = -(T_\mathrm{i} - T_e)/\tau_\mathrm{temp}$$

where evidently

$$K = \frac{1}{n\tau_\mathrm{temp}}.$$

2.3 Energy tensor equilibration

2.3.1 Isotropization

Recall temperature tensor evolution:

$$\partial_t \mathbb{T}_\mathrm{s} = (q_\mathrm{s}/m_\mathrm{s})2\mathrm{Sym}(\mathbb{T}_\mathrm{s} \times \mathbf{B}) + (\mathbb{R}_\mathrm{s} + \mathbb{Q}_\mathrm{s}^\mathrm{f} + \mathbb{Q}_\mathrm{s}^\mathrm{t})/n_\mathrm{s}.$$

A linear closure for $\mathbb{R}_\mathrm{s}$ is relaxation toward isotropy,

$$\mathbb{R}_\mathrm{s} = (p_\mathrm{s}\mathbb{I} - \mathbb{P}_\mathrm{s})/\tau_\mathrm{s}^\mathrm{c}, \quad \text{i.e.,} \quad \mathbb{R}_\mathrm{s}/n_\mathrm{s} = (T_\mathrm{s}\mathbb{I} - \mathbb{T}_\mathrm{s})/\tau_\mathrm{s}^\mathrm{c},$$

where $\tau_\mathrm{s}^\mathrm{c}$ is the self-collision time of species s, $p := \mathrm{tr}\mathbb{P}/3$ is scalar pressure, $T := \mathrm{tr}\mathbb{T}/3$ is scalar temperature, and $\mathbb{I}$ is the identity tensor. Braginskii says that

$$\tau_\mathrm{s}^\mathrm{c} = \tilde{\tau}_0 \frac{\sqrt{m_\mathrm{s}}}{n_\mathrm{s}} T_\mathrm{s}^{3/2}.$$

2.3.2 Thermal equilibration

For tensor thermal equilibration we can straightfowardly generalize scalar thermal equilibration:

$$\mathbb{Q}_\mathrm{i}^\mathrm{t} = K n_\mathrm{i} n_\mathrm{e}(\mathbb{T}_\mathrm{e} - \mathbb{T}_\mathrm{i}).$$

Then e.g. $\partial_t \mathbb{T}_\mathrm{i} = \frac{1}{n_\mathrm{i}} Q_\mathrm{s}^\mathrm{t}$ says $\partial_t \mathbb{T}_\mathrm{i} = K n_\mathrm{e}(\mathbb{T}_\mathrm{e} - \mathbb{T}_\mathrm{i})$, so as before $K = (n\tau_\mathrm{temp})^{-1}$.

2.3.3 Frictional heating

To generalize the scalar closure for frictional heating it is necessary to specify how to allocate resistive heating among directions parallel and perpendicular to the direction of relative motion of the two fluids.

Part B: solving

3 The electro-momentum system

The momentum equations in conjunction with the electric field equation constitute a linear system of equations with constant coefficients. We can divide out by density (since it is constant) and thus replace momentum evolution with velocity evolution. Neglecting the drag force (which would exponentially damp current) we get an ODE with constant coefficients:

$$\partial_t \begin{bmatrix} \mathbf{E} \\ \mathbf{u}_\mathrm{i} \\ \mathbf{u}_\mathrm{e} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{en_\mathrm{i}}{\varepsilon_0} & \frac{en_\mathrm{e}}{\varepsilon_0} \\ \frac{e}{m_i} & -\frac{e\mathbf{B}}{m_\mathrm{i}} \times \mathbb{I} & 0 \\ -\frac{e}{m_e} & 0 & \frac{e\mathbf{B}}{m_\mathrm{e}} \times \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{u}_\mathrm{i} \\ \mathbf{u}_\mathrm{e} \end{bmatrix}.$$

We can make this ODE antisymmetric by rescaling. For a generic rescaling, suppose

$$\mathbf{E} = \tilde{\mathbf{E}}\mathbf{E}_0,$$
$$\mathbf{u}_\mathrm{i} = \tilde{\mathbf{u}}_\mathrm{i}\mathbf{u}_{\mathrm{i}0},$$
$$\mathbf{u}_\mathrm{e} = \tilde{\mathbf{u}}_\mathrm{e}\mathbf{u}_{\mathrm{e}0}.$$

Making this substitution gives the system

$$\partial_t \begin{bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{u}}_\mathrm{i} \\ \tilde{\mathbf{u}}_\mathrm{e} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{en_\mathrm{i}}{\varepsilon_0}\frac{\mathbf{u}_{\mathrm{i}0}}{\mathbf{E}_0} & \frac{en_\mathrm{e}}{\varepsilon_0}\frac{\mathbf{u}_{\mathrm{e}0}}{\mathbf{E}_0} \\ \frac{e}{m_i}\frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{i}0}} & -\frac{e\mathbf{B}}{m_\mathrm{i}} \times \mathbb{I} & 0 \\ -\frac{e}{m_e}\frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{e}0}} & 0 & \frac{e\mathbf{B}}{m_\mathrm{e}} \times \mathbb{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{u}}_\mathrm{i} \\ \tilde{\mathbf{u}}_\mathrm{e} \end{bmatrix}.$$

If we require this system to be antisymmetric then

$$\frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{i}0}} = \sqrt{\frac{\rho_\mathrm{i}}{\varepsilon_0}}, \qquad \frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{e}0}} = \sqrt{\frac{\rho_\mathrm{e}}{\varepsilon_0}},$$

(where recall that $\rho_\mathrm{i} = m_\mathrm{i} n_\mathrm{i}$ and $\rho_\mathrm{e} = m_\mathrm{e} n_\mathrm{e}$) and the system becomes

$$\partial_t \begin{bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{u}}_\mathrm{i} \\ \tilde{\mathbf{u}}_\mathrm{e} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_\mathrm{i}\mathbb{I} & \Omega_\mathrm{e}\mathbb{I} \\ \Omega_\mathrm{i}\mathbb{I} & -\mathbf{B}_\mathrm{i} \times \mathbb{I} & 0 \\ -\Omega_\mathrm{e}\mathbb{I} & 0 & -\mathbf{B}_\mathrm{e} \times \mathbb{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{u}}_\mathrm{i} \\ \tilde{\mathbf{u}}_\mathrm{e} \end{bmatrix},$$

where each entry in the block matrix represents a 3×3 matrix and where

$$\Omega_\mathrm{i} = e\sqrt{\frac{n_\mathrm{i}}{\varepsilon_0 m_\mathrm{i}}} \qquad \text{and} \qquad \Omega_\mathrm{e} = e\sqrt{\frac{n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}}}$$

denote the ion and electron plasma frequencies and

$$\mathbf{B}_\mathrm{i} = \frac{e\mathbf{B}}{m_\mathrm{i}} \qquad \text{and} \qquad \mathbf{B}_\mathrm{e} = \frac{-e\mathbf{B}}{m_\mathrm{e}}$$

are the magnetic field rescaled for ions and electrons. Their magnitudes are the ion gyrofrequency $\omega_\mathrm{i} := |\mathbf{B}_\mathrm{i}|$ and the electron gyrofrequency $\omega_\mathrm{e} := |\mathbf{B}_\mathrm{e}|$.

3.1 Solution of perpendicular system

To solve the system we decompose into parallel and perpendicular components. Without loss of generality assume that **B** is in the direction of the third axis. Then our system decouples into a parallel system

$$\partial_t \begin{bmatrix} \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{u}}_{\mathrm{i}3} \\ \tilde{\mathbf{u}}_{\mathrm{e}3} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_\mathrm{i} & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & 0 & 0 \\ -\Omega_\mathrm{e} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{u}}_{\mathrm{i}3} \\ \tilde{\mathbf{u}}_{\mathrm{e}3} \end{bmatrix},$$

and a perpendicular system

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix}' = \begin{bmatrix} 0 & 0 & -\Omega_\mathrm{i} & 0 & \Omega_\mathrm{e} & 0 \\ 0 & 0 & 0 & -\Omega_\mathrm{i} & 0 & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & 0 & 0 & \omega_\mathrm{i} & 0 & 0 \\ 0 & \Omega_\mathrm{i} & -\omega_\mathrm{i} & 0 & 0 & 0 \\ -\Omega_\mathrm{e} & 0 & 0 & 0 & 0 & -\omega_\mathrm{e} \\ 0 & -\Omega_\mathrm{e} & 0 & 0 & \omega_\mathrm{e} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix}.$$

This is an antisymmetric matrix and therefore has imaginary eigenvalues and orthogonal eigenvectors. If we view the first and second components of each vector as real and imaginary parts, then this becomes a 3×3 complex linear differential equation with a skew hermitian coefficient matrix:

$$\partial_t \begin{bmatrix} \tilde{\mathbf{E}}_\perp \\ \tilde{\mathbf{u}}_{\mathrm{i}\perp} \\ \tilde{\mathbf{u}}_{\mathrm{e}\perp} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_\mathrm{i} & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & -i\omega_\mathrm{i} & 0 \\ -\Omega_\mathrm{e} & 0 & i\omega_\mathrm{e} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}}_\perp \\ \tilde{\mathbf{u}}_{\mathrm{i}\perp} \\ \tilde{\mathbf{u}}_{\mathrm{e}\perp} \end{bmatrix}, \qquad (1)$$

where we have used the natural isomorphism between $SO(2,\mathbb{R})$ and complex numbers

$$a+ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Observe that the parallel system is the special case of this system when the magnetic field is zero.

To generalize, suppose we want to solve the constant-coefficient linear ODE

$$\underline{x}' = \underline{\underline{A}} \cdot \underline{x}.$$

Seeking a solution $\underline{x}(t) = \underline{v}\exp(\lambda t)$ (where $\underline{v} \neq \underline{0}$) leads to the eigenvector problem

$$\underline{v}\lambda = \underline{\underline{A}} \cdot \underline{v}, \qquad \text{i.e.,} \qquad (\underline{\underline{A}} - \mathbb{I}\lambda)\cdot \underline{v} = 0.$$

We recall the theory of skew-Hermitian and Hermitian matrices. Since $\underline{\underline{A}}$ is skew-Hermitian (i.e. $\underline{\underline{A}}^* = -\underline{\underline{A}}$, where * denotes the conjugate of the transpose), $\underline{\underline{B}} := i\underline{\underline{A}}$ is Hermitian (i.e. $\underline{\underline{B}}^* = \underline{\underline{B}}$).

The eigenvalues of a Hermitian matrix are real. Indeed, assuming without loss of generality that $\underline{v}^*\underline{v} = 1$,

$$\begin{aligned} \lambda &= \underline{v}^*\underline{v}\lambda = \underline{v}^*\underline{\underline{B}}\underline{v} = \underline{v}^*\underline{\underline{B}}^*\underline{v} = (\underline{v}^*\underline{\underline{B}}\underline{v})^* = (\underline{v}^*\underline{v}\lambda)^* \\ &= \underline{v}^*\underline{v}\lambda^* = \lambda^*, \end{aligned}$$

and eigenvectors for different eigenvalues are orthogonal:

$$\begin{aligned} \underline{v}_2^*\underline{v}_1\lambda_1 &= \underline{v}_2^*\underline{\underline{B}}\underline{v}_1 = \underline{v}_2^*\underline{\underline{B}}^*\underline{v}_1 = (\underline{v}_1^*\underline{\underline{B}}\underline{v}_2)^* = (\underline{v}_1^*\underline{v}_2\lambda_2)^* \\ &= \underline{v}_2^*\underline{v}_1\lambda_2, \end{aligned}$$

which says that either $\underline{v}_2^*\underline{v}_1 = 0$ or $\lambda_1 = \lambda_2$.

Note that if $(\underline{v}, \omega)$ is an eigenvector-eigenvalue pair for $\underline{\underline{B}}$ then $(\underline{v}, i\omega)$ is an eigenvector-eigenvalue pair for $\underline{\underline{A}}$.

To find the eigenstructure we solve

$$0 = (\underline{\underline{A}} - i\omega)\cdot\underline{v} = \begin{bmatrix} -i\omega & -\Omega_\mathrm{i} & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & -i(\omega_\mathrm{i}+\omega) & 0 \\ -\Omega_\mathrm{e} & 0 & i(\omega_\mathrm{e}-\omega) \end{bmatrix}\cdot\underline{v}. \qquad (2)$$

If this has a nontrivial solution then the first row is a linear combination of the second two and we can ignore it. The second two equations then show that an eigenvector must be a multiple of the form

$$\underline{v} = \begin{bmatrix} i\beta_\mathrm{e}\beta_\mathrm{i} \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ \Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix}, \quad \text{where} \quad \beta_\mathrm{i} = \omega_\mathrm{i} + \omega \quad \text{and} \quad \beta_\mathrm{e} = \omega_\mathrm{e} - \omega,$$

as is confirmed (for the last two rows) by computing $(\underline{\underline{A}} - i\omega)\cdot\underline{v}$; the relation implied by the first row reveals the characteristic equation. Alternatively, the calculation

$$\begin{aligned} \underline{\underline{A}}\cdot\underline{v} &= \begin{bmatrix} 0 & -\Omega_\mathrm{i} & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & -i\omega_\mathrm{i} & 0 \\ -\Omega_\mathrm{e} & 0 & i\omega_\mathrm{e} \end{bmatrix}\cdot\begin{bmatrix} i\beta_\mathrm{e}\beta_\mathrm{i} \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ \Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix} \\ &= \begin{bmatrix} -\Omega_\mathrm{i}^2\beta_\mathrm{e} + \Omega_\mathrm{e}^2\beta_\mathrm{i} \\ i\beta_\mathrm{i}(\Omega_\mathrm{i}\beta_\mathrm{e}) - i\omega_\mathrm{i}(\Omega_\mathrm{i}\beta_\mathrm{e}) \\ -i(\Omega_\mathrm{e}\beta_\mathrm{i})\beta_\mathrm{e} + i\omega_\mathrm{e}(\Omega_\mathrm{e}\beta_\mathrm{i}) \end{bmatrix} = \underline{v}i\omega = \begin{bmatrix} i\beta_\mathrm{e}\beta_\mathrm{i} \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ \Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix} i\omega \end{aligned}$$

shows that ω must satisfy

$$\begin{aligned} &\beta_\mathrm{e}\beta_\mathrm{i}\omega = \Omega_\mathrm{i}^2\beta_\mathrm{e} - \Omega_\mathrm{e}^2\beta_\mathrm{i}, \\ &\omega = \beta_\mathrm{i} - \omega_\mathrm{i}, \\ &\omega = -\beta_\mathrm{e} + \omega_\mathrm{e}. \end{aligned}$$

The last two equations confirm that

$$\beta_\mathrm{i} = \omega_\mathrm{i} + \omega,$$
$$\beta_\mathrm{e} = \omega_\mathrm{e} - \omega,$$

and substituting these two relationships into the first equation gives the characteristic equation that an eigenvalue must satisfy:

$$(\omega_\mathrm{e} - \omega)(\omega_\mathrm{i} + \omega)\omega = \Omega_\mathrm{i}^2(\omega_\mathrm{e} - \omega) - \Omega_\mathrm{e}^2(\omega_\mathrm{i} + \omega).$$

Expanding in ω and collecting like terms gives

$$0 = \omega^3 + (\omega_\mathrm{i} - \omega_\mathrm{e})\omega^2 - (\omega_\mathrm{i}\omega_\mathrm{e} + \Omega_\mathrm{i}^2 + \Omega_\mathrm{e}^2)\omega \quad (3)$$
$$+ \cancelto{0}{(\Omega_\mathrm{i}^2\omega_\mathrm{e} - \Omega_\mathrm{e}^2\omega_\mathrm{i})}, \quad (4)$$

which is ω times a quadratic.

Note that the eigenvector $\underline{v} = \begin{bmatrix} i\beta_\mathrm{e}\beta_\mathrm{i} \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ \Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix}$ is never zero; indeed, Ω_i and Ω_e are strictly positive, and $\beta_\mathrm{i} = \omega_\mathrm{i} + \omega$ and $\beta_\mathrm{e} = \omega_\mathrm{e} - \omega$ cannot both be zero since otherwise $\omega_\mathrm{i} = -\omega$ and $\omega_\mathrm{e} = \omega$, contradicting that ω_i and ω_e are both strictly positive.

By the theory of Hermitian matrices a full set of orthogonal eigenvectors must exist. Since each eigenvector has a one-dimensional eigenspace, there must be three distinct eigenvalues ω.

Decompose $\underline{v}$ into real and imaginary parts:

$$\underline{v} = \underline{a} + i\underline{b} = \begin{bmatrix} 0 \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ \Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix} + i \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i} \\ 0 \\ 0 \end{bmatrix}.$$

Observe that the real and imaginary parts are orthogonal. Note that

$$-i\underline{v} = \underline{b} - i\underline{a} = \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i} \\ 0 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ \Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix}$$

is also an eigenvector with eigenvalue $i\omega$ and that these two eigenvectors are orthogonal: $\underline{v}^*(-i\underline{v}) = 2\underline{a} \cdot \underline{b} = 0$. The eigenvector-eigenvalue pair $(\underline{v}, i\omega)$ corresponds to the solution

$$\begin{aligned} \underline{v}\exp(i\omega t) &= (\underline{a} + i\underline{b})(\cos\omega t + i\sin\omega t) \\ &= (\underline{a}\cos\omega t - \underline{b}\sin\omega t) + i(\underline{b}\cos\omega t + \underline{a}\sin\omega t) \\ &= \begin{bmatrix} -\beta_\mathrm{e}\beta_\mathrm{i}\sin\omega t \\ \Omega_\mathrm{i}\beta_\mathrm{e}\cos\omega t \\ \Omega_\mathrm{e}\beta_\mathrm{i}\cos\omega t \end{bmatrix} + i \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i}\cos\omega t \\ \Omega_\mathrm{i}\beta_\mathrm{e}\sin\omega t \\ \Omega_\mathrm{e}\beta_\mathrm{i}\sin\omega t \end{bmatrix}, \end{aligned}$$

and the eigenvector-eigenvalue pair $(-i\underline{v}, i\omega)$ corresponds to the solution

$$\begin{aligned} -i\underline{v}\exp(i\omega t) &= (\underline{b} - i\underline{a})(\cos\omega t + i\sin\omega t) \\ &= (\underline{b}\cos\omega t + \underline{a}\sin\omega t) + i(\underline{b}\cos\omega t - \underline{a}\sin\omega t) \\ &= \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i}\cos\omega t \\ \Omega_\mathrm{i}\beta_\mathrm{e}\sin\omega t \\ \Omega_\mathrm{e}\beta_\mathrm{i}\sin\omega t \end{bmatrix} + i \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i}\sin\omega t \\ -\Omega_\mathrm{i}\beta_\mathrm{e}\cos\omega t \\ -\Omega_\mathrm{e}\beta_\mathrm{i}\cos\omega t \end{bmatrix}. \end{aligned}$$

Observe that in each of these solutions the ion and electron currents are in phase and the electric field is 90 degrees out of phase relative to them.

These two solutions are independent when interpreted (in $SO(2,\mathbb{R})$) as real solutions:

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} -\beta_\mathrm{e}\beta_\mathrm{i}\sin\omega t \\ \beta_\mathrm{e}\beta_\mathrm{i}\cos\omega t \\ \Omega_\mathrm{i}\beta_\mathrm{e}\cos\omega t \\ \Omega_\mathrm{i}\beta_\mathrm{e}\sin\omega t \\ \Omega_\mathrm{e}\beta_\mathrm{i}\cos\omega t \\ \Omega_\mathrm{e}\beta_\mathrm{i}\sin\omega t \end{bmatrix} \text{ and } \begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i}\cos\omega t \\ \beta_\mathrm{e}\beta_\mathrm{i}\sin\omega t \\ \Omega_\mathrm{i}\beta_\mathrm{e}\sin\omega t \\ -\Omega_\mathrm{i}\beta_\mathrm{e}\cos\omega t \\ \Omega_\mathrm{e}\beta_\mathrm{i}\sin\omega t \\ -\Omega_\mathrm{e}\beta_\mathrm{i}\cos\omega t \end{bmatrix}.$$

Evaluated at time 0 these solutions are

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} 0 \\ \beta_\mathrm{e}\beta_\mathrm{i} \\ \Omega_\mathrm{i}\beta_\mathrm{e} \\ 0 \\ \Omega_\mathrm{e}\beta_\mathrm{i} \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} \beta_\mathrm{e}\beta_\mathrm{i} \\ 0 \\ 0 \\ -\Omega_\mathrm{i}\beta_\mathrm{e} \\ 0 \\ -\Omega_\mathrm{e}\beta_\mathrm{i} \end{bmatrix}.$$

Note that orthogonality of complex solutions is equivalent to orthogonality of real solutions. So we have found three distinct imaginary eigenvalues and 6 orthogonal eigenvectors for the original 6×6 antisymmetric matrix.

3.2 Parallel system

The parallel system

$$\partial_t \begin{bmatrix} \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{u}}_{\mathrm{i}3} \\ \tilde{\mathbf{u}}_{\mathrm{e}3} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_\mathrm{i} & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & 0 & 0 \\ -\Omega_\mathrm{e} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{u}}_{\mathrm{i}3} \\ \tilde{\mathbf{u}}_{\mathrm{e}3} \end{bmatrix},$$

is the special, singular case of the perpendicular system (1) when the magnetic field is zero.

In this case the system (2) becomes

$$0 = (\underline{\underline{A}} - i\omega) \cdot \underline{v} = \begin{bmatrix} -i\omega & -\Omega_\mathrm{i} & \Omega_\mathrm{e} \\ \Omega_\mathrm{i} & -i\omega & 0 \\ -\Omega_\mathrm{e} & 0 & -i\omega \end{bmatrix} \cdot \underline{v}.$$

So eigenvalue/eigenvector pairs are

$$\omega = 0, \quad \underline{v} = \begin{bmatrix} 0 \\ \Omega_{\mathrm{e}} \\ \Omega_{\mathrm{i}} \end{bmatrix} \quad \text{and} \quad \omega = \pm\Omega_p, \quad \underline{v} = \begin{bmatrix} i\omega \\ \Omega_{\mathrm{i}} \\ -\Omega_{\mathrm{e}} \end{bmatrix},$$

where $\Omega_p := \sqrt{\Omega_{\mathrm{i}}^2 + \Omega_{\mathrm{e}}^2}$ is the plasma frequency.

To get real solutions we look at the real and imaginary parts of one of the complex-conjugate pair of solutions. Choose $\omega = \Omega_p$. Write

$$\underline{a} + i\underline{b} = \begin{bmatrix} 0 \\ \Omega_{\mathrm{i}} \\ -\Omega_{\mathrm{e}} \end{bmatrix} + i \begin{bmatrix} \Omega_p \\ 0 \\ 0 \end{bmatrix}.$$

Analogous to (3.1), the real and imaginary parts are real solutions:

$$\begin{aligned} \underline{v}\exp(i\Omega_p t) &= (\underline{a} + i\underline{b})(\cos\Omega_p t + i\sin\Omega_p t) \\ &= (\underline{a}\cos\Omega_p t - \underline{b}\sin\Omega_p t) + i(\underline{b}\cos\Omega_p t + \underline{a}\sin\Omega_p t) \\ &= \begin{bmatrix} -\Omega_p \sin\Omega_p t \\ \Omega_{\mathrm{i}}\cos\Omega_p t \\ -\Omega_{\mathrm{e}}\cos\Omega_p t \end{bmatrix} + i \begin{bmatrix} \Omega_p\cos\Omega_p t \\ \Omega_{\mathrm{i}}\sin\Omega_p t \\ -\Omega_{\mathrm{e}}\sin\Omega_p t \end{bmatrix}. \end{aligned}$$

So three orthogonal eigensolutions are

$$\begin{bmatrix} 0 \\ \Omega_{\mathrm{e}} \\ \Omega_{\mathrm{i}} \end{bmatrix}, \quad \begin{bmatrix} -\Omega_p \sin\Omega_p t \\ \Omega_{\mathrm{i}}\cos\Omega_p t \\ -\Omega_{\mathrm{e}}\cos\Omega_p t \end{bmatrix}, \quad \begin{bmatrix} \Omega_p\cos\Omega_p t \\ \Omega_{\mathrm{i}}\sin\Omega_p t \\ -\Omega_{\mathrm{e}}\sin\Omega_p t \end{bmatrix}.$$

Evaluated at time 0 these solutions are

$$\begin{bmatrix} 0 \\ \Omega_{\mathrm{e}} \\ \Omega_{\mathrm{i}} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \Omega_{\mathrm{i}} \\ -\Omega_{\mathrm{e}} \end{bmatrix}, \quad \begin{bmatrix} \Omega_p \\ 0 \\ 0 \end{bmatrix}.$$

Agreement with perpendicular system. If we take the limit as $|\mathbf{B}| \to 0$ in the perpendicular system we expect the solutions to decouple into solutions for $\left(\tilde{\mathbf{E}}_1, \tilde{\mathbf{u}}_{\mathrm{i}1}, \tilde{\mathbf{u}}_{\mathrm{e}1}\right)^{\mathrm{T}}$ and $\left(\tilde{\mathbf{E}}_2, \tilde{\mathbf{u}}_{\mathrm{i}2}, \tilde{\mathbf{u}}_{\mathrm{e}2}\right)^{\mathrm{T}}$ that agree with the solutions for the parallel system. As $|\mathbf{B}| \to 0$, $\omega_{\mathrm{i}} \to 0$ and $\omega_{\mathrm{e}} \to 0$ and so $\beta_{\mathrm{i}} \to \omega$ and $\beta_{\mathrm{e}} \to -\omega$. For the limiting eigenfrequency $\omega = \Omega_p$ the parallel solutions

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} -\beta_{\mathrm{e}}\beta_{\mathrm{i}}\sin\omega t \\ \beta_{\mathrm{e}}\beta_{\mathrm{i}}\cos\omega t \\ \Omega_{\mathrm{i}}\beta_{\mathrm{e}}\cos\omega t \\ \Omega_{\mathrm{i}}\beta_{\mathrm{e}}\sin\omega t \\ \Omega_{\mathrm{e}}\beta_{\mathrm{i}}\cos\omega t \\ \Omega_{\mathrm{e}}\beta_{\mathrm{i}}\sin\omega t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} \beta_{\mathrm{e}}\beta_{\mathrm{i}}\cos\omega t \\ \beta_{\mathrm{e}}\beta_{\mathrm{i}}\sin\omega t \\ \Omega_{\mathrm{i}}\beta_{\mathrm{e}}\sin\omega t \\ -\Omega_{\mathrm{i}}\beta_{\mathrm{e}}\cos\omega t \\ \Omega_{\mathrm{e}}\beta_{\mathrm{i}}\sin\omega t \\ -\Omega_{\mathrm{e}}\beta_{\mathrm{i}}\cos\omega t \end{bmatrix}$$

when divided by $\beta_{\mathrm{e}} = -\Omega_p$ become

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} -\Omega_p\sin\Omega_p t \\ \Omega_p\cos\Omega_p t \\ \Omega_{\mathrm{i}}\cos\Omega_p t \\ \Omega_{\mathrm{i}}\sin\Omega_p t \\ -\Omega_{\mathrm{e}}\cos\Omega_p t \\ -\Omega_{\mathrm{e}}\sin\Omega_p t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} \Omega_p\cos\Omega_p t \\ \Omega_p\sin\Omega_p t \\ \Omega_{\mathrm{i}}\sin\Omega_p t \\ -\Omega_{\mathrm{i}}\cos\Omega_p t \\ -\Omega_{\mathrm{e}}\sin\Omega_p t \\ \Omega_{\mathrm{e}}\cos\Omega_p t \end{bmatrix},$$

and for the limiting eigenfrequency $\omega = -\Omega_p$ the parallel solutions when divided by $\beta_{\mathrm{e}} = \Omega_p$ become

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} -\Omega_p\sin\Omega_p t \\ -\Omega_p\cos\Omega_p t \\ \Omega_{\mathrm{i}}\cos\Omega_p t \\ -\Omega_{\mathrm{i}}\sin\Omega_p t \\ -\Omega_{\mathrm{e}}\cos\Omega_p t \\ \Omega_{\mathrm{e}}\sin\Omega_p t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} -\Omega_p\cos\Omega_p t \\ \Omega_p\sin\Omega_p t \\ -\Omega_{\mathrm{i}}\sin\Omega_p t \\ -\Omega_{\mathrm{i}}\cos\Omega_p t \\ \Omega_{\mathrm{e}}\sin\Omega_p t \\ \Omega_{\mathrm{e}}\cos\Omega_p t \end{bmatrix}.$$

When projected onto axis 1 these solutions all agree with the second and third eigensolutions for the parallel component, and likewise for axis 2.

For the limiting eigenfrequency $\omega = 0$, $\beta_{\mathrm{e}} := \omega_{\mathrm{e}} + \omega$ and $\beta_{\mathrm{i}} := \omega_{\mathrm{i}} - \omega$ both go to zero (since ω_{e} and ω_{i} go to zero as $\mathbf{B}$ goes to zero). We may infer that $\beta_{\mathrm{e}}\beta_{\mathrm{i}}$ goes very quickly to zero. When ω is small $\cos\omega t \approx 1$ and $\sin\omega t \approx 0$. So for small magnetic field we expect

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{E}}_2 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{i}2} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}2} \end{bmatrix} = \begin{bmatrix} -\beta_{\mathrm{e}}\beta_{\mathrm{i}}\sin\omega t \\ \beta_{\mathrm{e}}\beta_{\mathrm{i}}\cos\omega t \\ \Omega_{\mathrm{i}}\beta_{\mathrm{e}}\cos\omega t \\ \Omega_{\mathrm{i}}\beta_{\mathrm{e}}\sin\omega t \\ \Omega_{\mathrm{e}}\beta_{\mathrm{i}}\cos\omega t \\ \Omega_{\mathrm{e}}\beta_{\mathrm{i}}\sin\omega t \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \Omega_{\mathrm{i}}\beta_{\mathrm{e}} \\ 0 \\ \Omega_{\mathrm{e}}\beta_{\mathrm{i}} \\ 0 \end{bmatrix},$$

which agrees with the direction of the expected limiting eigensolution

$$\begin{bmatrix} \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{u}}_{\mathrm{i}1} \\ \tilde{\mathbf{u}}_{\mathrm{e}1} \end{bmatrix} \approx \begin{bmatrix} 0 \\ \Omega_{\mathrm{e}} \\ \Omega_{\mathrm{i}} \end{bmatrix}$$

if $\frac{\beta_{\mathrm{e}}}{\beta_{\mathrm{i}}} \to \left(\frac{\Omega_{\mathrm{e}}}{\Omega_{\mathrm{i}}}\right)^2$ as $\mathbf{B} \to 0$. I do not see how to show this in general, but in the neutral case where $n_{\mathrm{i}} = n_{\mathrm{e}}$, $\Omega_{\mathrm{i}}^2\omega_{\mathrm{e}} = \Omega_{\mathrm{e}}^2\omega_{\mathrm{i}}$, so the constant term vanishes in the characteristic polynomial equation (3), $\omega = 0$ is always an eigenvalue, and $\beta_{\mathrm{i}} = \omega_{\mathrm{i}}$ and $\beta_{\mathrm{e}} = \omega_{\mathrm{e}}$, so $\frac{\beta_{\mathrm{e}}}{\beta_{\mathrm{i}}} = \left(\frac{\Omega_{\mathrm{e}}}{\Omega_{\mathrm{i}}}\right)^2$ as needed.

4 The pressure tensor system

Ignoring interspecies collisions, the pressure tensor evolution equation with linear closure is

$$\partial_t \mathbb{P}_s = (q_s/m_s)2\mathrm{Sym}(\mathbb{P}_s \times \mathbf{B}) + \mathbb{R}_s,$$

where

$$\mathbb{R}_s = (\mathrm{tr}(\mathbb{P}_s)\mathbb{I} - \mathbb{P}_s)/\tau_s^c$$

and

$$\tau_s^c = \tilde{\tau}_0 \frac{\sqrt{m_s}}{n_s} T_s^{3/2}.$$

Observe that temperature isotropization leaves temperature invariant. So this is a linear ODE with constant coefficients. The $\mathbb{P} \times \mathbf{B}$ term rotates the pressure tensor around the magnetic field. The relaxation term relaxes the pressure toward isotropy. These two operations commute. So we can trivially solve this ODE exactly.

4.1 Rotation of the pressure tensor

The $\mathbb{P} \times \mathbf{B}$ term rotates the pressure tensor around the magnetic field vector. The rate of rotation is the species gyrofrequency, so the angle of rotation of the ion pressure tensor in time interval $\mathrm{d}t$ is $\omega_i \mathrm{d}t$.

Let $\mathbf{e}_i$ denote the ith standard basis vector. Let $\mathbf{e}_i'(t)$ denote the rotated version of $\mathbf{e}_i$. Let $\mathbb{P}(t)$ denote the rotated pressure tensor. The pressure tensor components are $\mathbb{P}_{mn}(t) = \mathbf{e}_m \cdot \mathbb{P} \cdot \mathbf{e}_n$. The pressure tensor components are invariant in a rotating (primed) coordinate frame:

$$\mathbb{P}(t) = \mathbb{P}_{ij}(0)\mathbf{e}_i'\mathbf{e}_j'.$$

Therefore, the components in the standard basis are:

$$\mathbb{P}_{mn}(t) = \mathbb{P}_{ij}(0)(\mathbf{e}_i' \cdot \mathbf{e}_m)(\mathbf{e}_j' \cdot \mathbf{e}_n).$$

Thus, to evolve the pressure tensor $\mathbb{P}_s$ for species s over a time interval $\mathrm{d}t$, we need to apply to the standard basis vectors a rotation with rotation vector $\mathbf{R} := \frac{q_s}{m_s}\mathbf{B}\,\mathrm{d}t$, i.e. with direction $\hat{\mathbf{b}} := \mathbf{B}/|\mathbf{B}|$ and angle $\theta := \omega_s\,\mathrm{d}t$, where $\omega_s := |\mathbf{B}|\frac{q_s}{m_s}$ is the gyrofrequency of species s.

To rotate a vector $\mathbf{u}$ by the vector $\mathbf{R} = \theta\hat{\mathbf{b}}$, where $\theta = |\mathbf{R}|$, decompose it into parallel and perpendicular components and rotate the perpendicular component:

$$\begin{aligned}
\mathbf{u}_\parallel &= \mathbf{u} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} \\
\mathbf{u}_\perp &= \mathbf{u} - \mathbf{u}_\parallel \\
\mathbf{u} &= \mathbf{u}_\parallel + \mathbf{u}_\perp
\end{aligned}$$

The rotated vector is

$$\begin{aligned}
\mathbf{u}' &= \mathbf{u}_\parallel + (\cos\theta)\mathbf{u}_\perp + (\sin\theta)\mathbf{u} \times \hat{\mathbf{b}}. \\
&= \mathbf{u}(\cos\theta) + (1 - \cos\theta)\mathbf{u}_\parallel + (\sin\theta)\mathbf{u} \times \hat{\mathbf{b}}
\end{aligned}$$

We can avoid renormalizing the rotation vector if we use the sine cardinal function.

$$\begin{aligned}
\mathbf{u}' = \mathbf{u}(\cos\theta) &+ 2\left(\sin\frac{\theta}{2}\right)^2 \mathbf{u} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} \\
&+ 2\left(\cos\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2}\right)\mathbf{u} \times \hat{\mathbf{b}} \\
= \mathbf{u}(\cos\theta) &+ \frac{1}{2}\left(\mathrm{sinc}\frac{\theta}{2}\right)^2 \mathbf{u} \cdot \mathbf{R}\mathbf{R} \\
&+ \left(\cos\frac{\theta}{2}\right)\left(\mathrm{sinc}\frac{\theta}{2}\right)\mathbf{u} \times \mathbf{R}
\end{aligned}$$

Recall that $\mathbf{e}_i'$ is the rotated version of the elementary basis vector $\mathbf{e}_i$. To express the components of the rotation matrix $\mathbb{R}_{ij} := \mathbf{e}_i \cdot \mathbf{e}_j'$, adopt the abbreviations $c := \cos\theta$ and $s := \sin\theta$. The rotated vector is

$$\begin{aligned}
\mathbf{e}_j' &= c\mathbf{e}_j + (1-c)\mathbf{e}_j \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} + s\mathbf{e}_j \times \hat{\mathbf{b}} \\
&= \mathbf{e}_j \cdot \underbrace{\left(c\mathbb{I} + (1-c)\hat{\mathbf{b}}\hat{\mathbf{b}} + s\mathbb{I} \times \hat{\mathbf{b}}\right)}_{\mathbb{R}^{\mathrm{T}}}
\end{aligned}$$

To determine the components of $\mathbb{I} \times \hat{\mathbf{b}}$, match up the identity

$$\mathbf{u} \times \hat{\mathbf{b}} = \mathbf{u} \cdot \mathbb{I} \times \hat{\mathbf{b}} = (\mathbb{I} \times \hat{\mathbf{b}})^{\mathrm{T}} \cdot \mathbf{u}$$

with the coordinate expansion

$$\begin{aligned}
\mathbf{u} \times \hat{\mathbf{b}} &= \begin{pmatrix} u_2 b_3 - u_3 b_2 \\ u_3 b_1 - u_1 b_3 \\ u_1 b_2 - u_2 b_1 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}}_{(\mathbb{I}\times\hat{\mathbf{b}})^{\mathrm{T}}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}
\end{aligned}$$

So the rotation matrix $\mathbb{R}$ is:

$$\begin{bmatrix}
b_1 b_1(1-c) + c & b_1 b_2(1-c) + s b_3 & b_1 b_3(1-c) - s b_2 \\
b_2 b_1(1-c) - s b_3 & b_2 b_2(1-c) + c & b_2 b_3(1-c) + s b_1 \\
b_3 b_1(1-c) + s b_2 & b_3 b_2(1-c) - s b_1 & b_3 b_3(1-c) + c
\end{bmatrix},$$

where we are free to make the replacements

$$\begin{aligned}(1-c)b_i b_j &= \quad (1/2)\,\mathrm{sinc}^2(\theta/2) R_i R_j,\\ s b_i \quad &= \mathrm{sinc}(\theta/2)\cos(\theta/2) R_i.\end{aligned}$$

5 Positivity

An exact solver for the source term can be used as a component of a positivity-preserving discontinuous Galerkin solver for the two-fluid equations. This requires use of time-splitting to handle the source term and the flux separately.

Positivity-preserving methods work by maintaining two conditions: (1) the average state in each mesh cell satisfies positivity, and (2) the state at a set of positivity points satisfies positivity. At the beginning of a time step positivity of each cell average is assumed and positivity is enforced at the cell's positivity points by if necessary rescaling the perturbation by a damping factor just sufficiently smaller than 1.

For a source-term time step, the positivity points are simply the Gaussian quadrature points used to integrate over the cell volume. The exact source term solver samples the state at each Gaussian quadrature point, solves the source term exactly there, and then uses Gaussian quadrature to project onto the modal representation of the cell averages (sampling and projection are no-op's in the case of nodal DG). The cell average is a convex combination of the state at the quadrature points, so positivity of the cell average is maintained because density and the minimal eigenpressure are (not necessarily strictly) concave functions of the state variables.

I remark that use of an exact solver for the source term also allows a ten-moment solver to conserve energy exactly.

References

[1] Harish Kumar, Finite volume methods for the two-fluid MHD equations, Hyp 2010 Beijing (Also as SAM Report, see `http://www.sam.math.ethz.ch/reports/2010/29`).