Taylor Series

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Theorem 1 (Taylor, with integral remainder)**.** *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *have* $n+1$ *continuous derivatives on the interval* $[a, x]$*. Then*

$$\begin{aligned} f(x) = &\sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n \\ =&f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) \\ &+ \cdots + \frac{(x-a)^k}{k!} f^(k)(a) \\ &+ \cdots + \frac{(x-a)^n}{n!} f^n(a) + R_n, \end{aligned}$$

where

$$R_n = \int_a^x \frac{(x-s)^n}{n!} f^{(n+1)}(s)\, ds.$$

Corollary 2 (Lagrange remainder)**.**

$$\exists c \in [a, x] \text{ such that } R_n = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

Proof of Theorem 1. To prove Taylor's theorem, we begin with the fundamental theorem of Calculus in the form

$$f(x) = f(a) + \underbrace{\int_a^x f'(s)ds}_{R_0},$$

which is just Taylor's theorem with integral remainder for $n = 0$.

To derive Taylor's theorm, we will make use of integration by parts, which says that for any functions g, h differentiable on $[a, x]$,

$$\int_a^x h'g = [hg]_a^x - \int_a^x hg'.$$

In deriving Taylor's theorem, it is convenient to replace h with $-h$ and rewrite this as

$$\int_a^x (-h)'g = [hg]_x^a + \int_a^x hg'.$$

We wish to express $f(x)$ in terms of the value of its derivatives at the a boundary of the interval $[a, x]$. So we view the integrand as $1 \cdot f'$ and use integration by parts to transfer the derivative from 1 to f'. $\int 1ds = s - C$; we will choose C to eliminate the term from the x boundary of $[a, x]$:

$$\begin{aligned} R_0 &= \int_a^x f'(s)ds \\ &= [(s-C)f'(s)]_{s=a}^x - \int_a^x (s-C)f''(s)ds \\ &= [(C-s)f'(s)]_{s=x}^a + \int_a^x (C-s)f''(s)ds \end{aligned}$$

To eliminate the $f'(x)$ term, we choose $C = x$. Then

$$R_0 = (x-a)f'(a) + \underbrace{\int_a^x (x-s)f''(s)ds,}_{R_1}$$

which shows Taylor's theorem for $n = 1$.

To prove Taylor's theorem with integral remainder in general, it is enough to show that

$$R_{n-1} = \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n. \tag{1}$$

(This is a disguised proof by induction.) Indeed,

$$\begin{aligned} R_{n-1} =& \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} f^{(n)}(s)\, ds \\ =&\Big[\frac{(x-s)^n}{n!} f^{(n)}(s)\Big]_{s=x}^a + \int_a^x \frac{(x-s)^n}{n!} f^{(n+1)}(s)\, ds \\ =&\frac{(x-a)^n}{n!} f^{(n)}(a) + R_n, \end{aligned}$$

as needed. ∎

Proof of Corollary 2. Let $m = \min_{s \in [a,x]} f^{(n+1)}(s)$ and let $M = \max_{s \in [a,x]} f^{(n+1)}(s)$. Then the image of the interval $[a, x]$ under the continuous function $f^{(n+1)}$ is $f^{(n+1)}([a, x]) = [m, M]$, so

$$\begin{aligned} R_n &\in \int_a^x \frac{(x-s)^{n+1}}{n!} [m, M] ds \\ &= \frac{(x-a)^{n+1}}{(n+1)!} [m, M] \\ &= \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}([a, x]), \end{aligned}$$

i.e. $\exists c \in [a, x]$ such that $R_n = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$, as needed. ∎

1 Multiple variables.

Corollary 3 (Taylor for multiple variables)**.** *Let* $f : \mathbb{R}^m \mapsto \mathbb{R}$ *have* $n+1$ *continuous partial derivatives on an open region containing the interval* $[\mathbf{r}_0, \mathbf{r}_1]$*. Then*

$$f(\mathbf{r}_1) = \sum_{k=0}^{n} \frac{((\mathbf{r}_1 - \mathbf{r}_0) \cdot \nabla)^k f(\mathbf{r}_0)}{k!} + \frac{((\mathbf{r}_1 - \mathbf{r}_0) \cdot \nabla)^{n+1} f(\mathbf{r}_c)}{(n+1)!},$$

for some $\mathbf{r}_c$ *on the line segment between* $\mathbf{r}_0$ *and* $\mathbf{r}_1$*.*

Proof of Corollary 3. Let $h(t) = f(\mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0))$. Then for some $c \in [0, 1]$

$$\begin{aligned} f(\mathbf{r}_1) =& h(1) = \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} + \frac{h^{(n+1)}(c)}{(n+1)!} \\ =& \sum_{k=0}^{n} \frac{((\mathbf{r}_1 - \mathbf{r}_0) \cdot \nabla)^k f(\mathbf{r}_0)}{k!} + \frac{((\mathbf{r}_1 - \mathbf{r}_0) \cdot \nabla)^{n+1} f(\mathbf{r}_c)}{(n+1)!}, \end{aligned}$$

where $\mathbf{r}_c := \mathbf{r}_0 + c(\mathbf{r}_1 - \mathbf{r}_0)$ lies on the line segment between $\mathbf{r}_0$ and $\mathbf{r}_1$. □

References

[1] `http://en.wikipedia.org/wiki/Taylor_series`

[Rudin53] Walter Rudin, *Principles of Mathematical Analysis,* McGraw-Hill, ©1953.