Ten-moment two-species closure
by Alec Johnson, January 2010

We first briefly recount the derivation of the 10-moment system. This development is given in greater detail and generality in my note *General Moment Evolution*.

1 Boltzmann equation.

The Boltzmann equation evolves particle mass density functions $f_s(\mathbf{x}, \mathbf{v}, t)$ (of position, particle velocity, and time) for each species s, where $f_s|d^3\mathbf{x}\wedge d^3\mathbf{v}|$ is the amount of mass of species s in the infinitesimal phase space volume $|d^3\mathbf{x} \wedge d^3\mathbf{v}|$. (We will consider two-species hydrogen plasmas; i and e will denote ion and electron species indices.) The Boltzmann equation asserts conservation (or balance) of particles in phase space,

$$\partial_t f_s + \nabla_\mathbf{x} \cdot (\mathbf{v} f_s) + \nabla_\mathbf{v} \cdot \left(\frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) f_s \right) = \mathcal{C}_s,$$

where $q_s = \pm e$ is particle charge, m_s is particle mass, $\mathbf{E}$ is electric field, $\mathbf{B}$ is magnetic field, and $\mathcal{C}_s$ is the collision operator.

The collision operator is the sum of collision operators representing interaction with each species:

$$\mathcal{C}_s = \widetilde{\mathcal{C}}_s + \overleftrightarrow{\mathcal{C}}_s,$$

where $\widetilde{\mathcal{C}}_s$, an intraspecies collision operator, is a function of $\mathbf{v} \mapsto f_s(t, \mathbf{x}, \mathbf{v})$, and where

$$\overleftrightarrow{\mathcal{C}}_s := \sum_{p \neq s} \overleftrightarrow{\mathcal{C}}_{sp}$$

represents the net affect of all other species p; $\overleftrightarrow{\mathcal{C}}_{sp}$ is an interspecies collision operator which represents the affect on species s of collisions with species p and is a function of $\mathbf{v} \mapsto f_s(t, \mathbf{x}, \mathbf{v})$ and $\mathbf{v} \mapsto f_p(t, \mathbf{x}, \mathbf{v})$. We adopt the conventions that a bidirectional arrow over a symbol indicates interaction between two different species and that when two indices are shown the first index indicates the species acted upon.

The Boltzmann equation is coupled to **Maxwell's equations**,

$$\begin{aligned} \partial_t \mathbf{B} &= -\nabla \times \mathbf{E}, & \nabla \cdot \mathbf{B} &= 0, \\ \partial_t \mathbf{E} &= c^2 \nabla \times B - \mathbf{J}/\epsilon_0, & \nabla \cdot \mathbf{E} &= \sigma/\epsilon_0, \end{aligned}$$

where c is the speed of light, ϵ_0 is the permittivity constant, the net charge density σ is the sum of the charge densities $\sigma_s := q_s \int_\mathbf{v} f_s$ of each species, and the net current density $\mathbf{J}$ is the sum of the current densities $\mathbf{J}_s := q_s \int_\mathbf{v} f_s \mathbf{v}$ of each species.

We will henceforth assume a default species index s and generally will use an explicit species index only when when referring to interaction with another species p.

2 Generic 10-moment model

2.1 "Conserved" variables. Multiplying the Boltzmann equation by powers of $\mathbf{v}$ and integrating over all $\mathbf{v}$ yields generic gas-dynamic equations for the 10-moment model. For each species the zeroth moment ($\mathbf{v}^0 = 1$) asserts conservation of mass,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0;$$

and (dividing by particle mass m) conservation of particle density,

$$\partial_t n + \nabla \cdot (n \mathbf{u}) = 0;$$

the first moment ($\mathbf{v}^1 = \mathbf{v}$) asserts balance of momentum

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \mathbb{E} = \frac{q}{m} \rho (\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{R};$$

and the second moment ($\mathbf{v}^2 := \mathbf{v}\mathbf{v} := \mathbf{v} \otimes \mathbf{v}$) yields energy tensor evolution,

$$\begin{aligned} &\partial_t \mathbb{E} + 3\nabla \cdot \mathrm{Sym}(\mathbf{u}\mathbb{E}) - 2\nabla \cdot (\rho \mathbf{u}\mathbf{u}\mathbf{u}) + \nabla \cdot \mathbb{q} \\ &= \frac{q}{m} 2\,\mathrm{Sym}(\rho \mathbf{u}\mathbf{E} + \mathbb{E} \times \mathbf{B}) + 2\,\mathrm{Sym}(\mathbf{u}\mathbf{R}) + \mathbb{Q}, \end{aligned}$$

where the tensor symmetrization operator Sym denotes the average over all permutations of subscripts of its argument tensor; the variables in these gas-dynamic equations are defined as follows: $\rho := \int_\mathbf{v} f$ is is mass density, $n = \rho/m_s$ is particle density, $\mathbf{u} = \int_\mathbf{v} f \mathbf{v}/\rho$ is bulk fluid velocity, $\mathbf{c} := \mathbf{v} - \mathbf{u}$ is thermal particle velocity, $\mathbb{E} := \int_\mathbf{v} f \mathbf{v}\mathbf{v}$ is the energy tensor, $\mathbb{q} := \int_\mathbf{v} f \mathbf{c}\mathbf{c}\mathbf{c}$ is generalized heat flux, $\mathbf{R} := \int_\mathbf{v} \mathcal{C}\mathbf{c}$ is resistive drag force on species s, and $\mathbb{Q} := \int_\mathbf{v} \mathcal{C}\mathbf{c}\mathbf{c}$ is the collisional thermal energy source.

It is convenient to decompose the resistive drag force and the collisional thermal energy into contributions from each species:

$$\mathbf{R} = \sum_{p \neq s} \overleftrightarrow{\mathbf{R}}_p, \qquad \mathbb{Q} = \mathbb{R} + \sum_{p \neq s} \overleftrightarrow{\mathbb{Q}}_p;$$

here $\overleftrightarrow{\mathbf{R}}_{\mathrm{p}} := \int_{\mathbf{v}} \overleftrightarrow{\mathcal{C}}_{\mathrm{p}}\mathbf{c}$ represents resistive drag force from species p, $\overleftrightarrow{\mathbb{Q}}_{\mathrm{p}} := \int_{\mathbf{v}} \overleftrightarrow{\mathcal{C}}_{\mathrm{p}}\mathbf{cc}$ represents generalized heating due to collisions with species p, and $\mathbb{R} := \int_{\mathbf{v}} \widetilde{\mathcal{C}}\mathbf{cc}$ represents pressure change (typically relaxation toward isotropic pressure) due to intraspecies collisions. For intraspecies collisions, conservation of momentum says $\widetilde{\mathbf{R}} := \int_{\mathbf{v}} \widetilde{\mathcal{C}}\mathbf{c} = 0$. Also, in case collisions involve no exchange of energy with nontranslational modes, conservation of energy says $\operatorname{tr} \mathbb{R} = \int_{\mathbf{v}} \widetilde{\mathcal{C}} c^2 = 0$.

These generic balance laws do not constitute a closed system; we need to specify the heat fluxes $\mathbb{q}$, the resistive drag forces $\mathbf{R}$, and the collisional thermal sources $\mathbb{Q}$ in terms of the evolved state variables. To this end we now transform to primitive variables.

2.2 Pressure (thermal energy) tensor evolution. Physical laws should be invariant under change of reference frame, so we express moment evolution equations for second-order and higher moments in terms of primitive variables (which are defined relative to the center of mass velocity) rather than conserved variables when we seek closure relations. More generally, as is necessary in the relativistic case, one seeks closure specifically by first transforming the system into the *reference frame of the fluid*.

The energy tensor may be written as the sum of a kinetic energy tensor and a thermal energy tensor: $\mathbb{E} = \rho\mathbf{uu} + \mathbb{P}$, where the thermal energy tensor $\mathbb{P} := \int_{\mathbf{v}} f\mathbf{cc}$ is also known as the pressure tensor. For the nonrelativistic 10-moment system transforming to primitive variables means replacing the evolution equation for the energy tensor with an evolution equation for the pressure tensor. To get the evolution equation for the pressure tensor you can multiply the Boltzmann equation by $\mathbf{cc}$ and integrate over all velocities. Alternatively, you can multiply the momentum equation by bulk fluid velocity to get a kinetic energy tensor evolution equation and subtract it from the energy tensor evolution equation.

In terms of the pressure tensor the energy tensor evolution equation can be written

$$\begin{aligned}&\partial_t \mathbb{E} + \nabla\cdot(\mathbf{u}\mathbb{E}) + 2\operatorname{Sym}\nabla\cdot(\mathbb{P}\mathbf{u}) + \nabla\cdot\mathbb{q}\\ &= \frac{q}{m} 2\operatorname{Sym}(\rho\mathbf{uE} + \mathbb{E}\times\mathbf{B}) + 2\operatorname{Sym}(\mathbf{uR}) + \mathbb{Q}.\end{aligned}$$

Multiplying the momentum evolution equation by $2\mathbf{u}$, taking the symmetric part, and assuming a smooth solution gives the kinetic energy tensor evolution equation

$$\begin{aligned}&\partial_t(\rho\mathbf{uu}) + \nabla\cdot(\rho\mathbf{uuu}) + 2\operatorname{Sym}(\mathbf{u}\nabla\cdot\mathbb{P})\\ &= \frac{q}{m} 2\operatorname{Sym}(\rho\mathbf{uE} + \rho\mathbf{uu}\times\mathbf{B}) + 2\operatorname{Sym}(\mathbf{uR}).\end{aligned}$$

Subtracting the kinetic energy tensor evolution equation from the energy evolution equation gives the pressure tensor evolution equation

$$\begin{aligned}&\partial_t\mathbb{P} + \nabla\cdot(\mathbf{u}\mathbb{P}) + 2\operatorname{Sym}(\mathbb{P}\cdot\nabla\mathbf{u}) + \nabla\cdot\mathbb{q}\\ &= \frac{q}{m} 2\operatorname{Sym}(\mathbb{P}\times\mathbf{B}) + \mathbb{Q}\end{aligned}$$

(where we have used that $2\operatorname{Sym}(\nabla\cdot(\mathbb{P}\mathbf{u}) - \mathbf{u}\nabla\cdot\mathbb{P}) = 2\operatorname{Sym}(\mathbb{P}\cdot\nabla\mathbf{u})$).

2.3 Temperature evolution. A way to obtain closure is to assume the form of the distribution of particle density as a function of velocity and then evaluate the moments of the collision integral in terms of the parameters of the particle density distribution. We do not take this approach here; instead, we will seek a definition of entropy such that entropy is conserved for smooth flow in the absence of diffusive terms (heat flux and collisional thermal sources). We will then impose the requirement that the collisional terms may only allow entropy to increase; in combination with the requirement that the closure be isotropic and linear this will reveal the form of the closure.

A related generic procedure that avoids explicit evaluation of collision integrals is to determine local thermodynamic equlibrium in the reference frame of the fluid; one then posits that collisions relax the state of the fluid toward the equilibrium state. Since thermodynamic equilibrium maximizes entropy and relaxation toward equilibrium increases entropy, this procedure may be viewed as underlying the procedure we employ.

The requirement that entropy increase is essentially a requirement that the closure yield a well-posed partial differential equation without any source terms that would cause exponential growth; that is, there should be no antidiffusive terms and no exponential growth.

In the absence of heat flow and collisional source terms we expect entropy to be preserved along particle paths. Entropy is a thermodynamic concept, and thermodynamics is the study of equilibrium, so the entropy should be a function of state variables independent of reference frame and without reference to temporal or spatial derivatives or frame of reference. So neglecting $\mathbb{q}$ and $\mathbb{Q}$ we seek to write the pressure evolution equation in the form $d_t s = 0$ for some appropriately defined s which should represent the entropy per mass.

Hence we first write mass conservation and pressure tensor evolution as material derivatives. Conservation of mass becomes

$$d_t\rho + \rho\nabla\cdot\mathbf{u} = 0;$$

Solved for $\nabla\cdot\mathbf{u}$ conservation of mass says that the divergence of the fluid velocity is minus the convective logarithmic derivative of the mass (or particle) density:

$$\nabla\cdot\mathbf{u} = -\rho^{-1}d_t\rho = -d_t\ln\rho = -d_t\ln n.$$

To rewrite pressure evolution in terms of the convective derivative we first write it in terms of the "bulk derivative" $\eth_t$, defined by

$$\eth_t\alpha := \partial_t\alpha + \nabla\cdot(\mathbf{u}\alpha) = (d_t + \nabla\cdot\mathbf{u})\alpha \quad (\forall\alpha).$$

Then:

$$\begin{aligned}&\eth_t\mathbb{P} + 2\,\mathrm{Sym}(\mathbb{P}\cdot\nabla\mathbf{u}) + \nabla\cdot\mathbb{q}\\ &= \frac{q}{m}2\,\mathrm{Sym}(\mathbb{P}\times\mathbf{B}) + \mathbb{Q}.\end{aligned}$$

Conservation of mass ($\eth_t\rho = 0$) implies that the bulk derivative of a density per volume is the volume density of the convective derivative of density per mass:

$$\boxed{\eth_t(\rho\beta) = \rho d_t\beta \quad (\forall\beta)}.$$

So to get a simple equation with a convective derivative we define the **temperature tensor** to be the pressure tensor divided by the number density, i.e.,

$$\mathbb{T} := \mathbb{P}/n.$$

(Since he is dealing with only one species, Levermore (see references) instead defines the pseudo temperature tensor $\Theta := \mathbb{P}/\rho = \mathbb{T}/m$.) Substituting into the pressure tensor evolution equation gives us temperature tensor evolution,

$$\begin{aligned}&nd_t\mathbb{T} + 2n\,\mathrm{Sym}(\mathbb{T}\cdot\nabla\mathbf{u}) + \nabla\cdot\mathbb{q}\\ &= \frac{q}{m}2n\,\mathrm{Sym}(\mathbb{T}\times\mathbf{B}) + \mathbb{Q}.\end{aligned}$$

3 Generic 5-moment model

Every tensor equation that has appeared so far in this note has a corresponding scalar equation obtained by taking (half) its trace. Therefore for inspiration we derive entropy evolution for the scalar case, taking care to choose steps that generalize straightforwardly to matrices. The generalization will require us to delay taking the trace.

To ensure that an isotropic pressure tensor is the scalar pressure times the identity matrix, the scalar pressure is defined to be one third the trace of the pressure tensor, and likewise for the temperature:

$$\begin{aligned}&p := \mathrm{tr}\ \mathbb{P}/3,\\ &T := \mathrm{tr}\ \mathbb{T}/3 = p/n.\end{aligned}$$

(Levermore instead defines the pseudo temperature $\theta := \mathrm{tr}\ \Theta/3 = p/\rho$.) Observe that $p = nT$. The energy density is defined to be half the trace of the energy tensor,

$$\mathcal{E} := \mathrm{tr}\ \mathbb{E}/2.$$

Half the trace of the relation $\mathbb{E} = \mathbb{P} + \rho\mathbf{uu}$ yields

$$\mathcal{E} = \frac{3}{2}p + \rho u^2/2.$$

We now take the trace of the tensor equations seen so far. Energy tensor evolution yields scalar energy evolution,

$$\begin{aligned}&\partial_t\mathcal{E} + \nabla\cdot(\mathbf{u}\mathcal{E} + \mathbf{u}\cdot\mathbb{P}) + \nabla\cdot\mathbf{q}\\ &= \mathbf{J}\cdot\mathbf{E} + \mathbf{u}\cdot\mathbf{R} + Q,\end{aligned}$$

where the heat flux vector is defined to be half the trace of the heat flux tensor, $2\mathbf{q} := \mathrm{tr}\ \mathbb{q} = \int_{\mathbf{v}} fc^2\mathbf{c}$, and the collisional heating term is half the trace of the collisional thermal energy tensor source, $2Q := \mathrm{tr}\ \mathbb{Q} = \int_{\mathbf{v}}\mathcal{C}c^2$. Half the trace of the decomposition of the collisional thermal energy tensor source yields the scalar decomposition of the heat source,

$$Q = \widetilde{Q} + \sum_{\mathrm{p}\neq\mathrm{s}} \overleftrightarrow{Q}_{\mathrm{p}},$$

where $2\widetilde{Q} := \mathrm{tr}(\mathbb{R})$ and $2\overleftrightarrow{Q}_{\mathrm{p}} := \mathrm{tr}(\overleftrightarrow{\mathbb{Q}}_{\mathrm{p}})$.

Half the trace of pressure tensor evolution yields thermal energy evolution (i.e. scalar pressure evolution),

$$3\delta_t p/2 + p\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{q} = \boldsymbol{\sigma} : \nabla\mathbf{u} + Q,$$

where we have decomposed the pressure tensor into a scalar pressure p and an anticipated viscous stress tensor, $\mathbb{P} =: p\mathbb{I} - \boldsymbol{\sigma}$, where $\mathbb{I}$ is the identity tensor. Substituting $p = nT$ or taking half the trace of temperature tensor evolution gives thermal energy evolution in terms of scalar temperature,

$$n3d_t T/2 + nT\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{q} = \boldsymbol{\sigma} : \nabla\mathbf{u} + Q.$$

4 Derivation of entropy for the 5-moment model (scalar energy, pressure, and temperature)

Basic thermodynamics says that the differential change in entropy (of a component of a system in thermal equilibrium) is the differential heat absorbed divided by the temperature of the system,

$$dS = \frac{dQ}{T}.$$

Multiplying the thermal energy evolution equation by T^{-1} (and force-factoring out n) gives

$$\begin{aligned} &n(3d_t \ln T/2 - d_t \ln n) + T^{-1}\nabla\cdot\mathbf{q} \\ &= T^{-1}\boldsymbol{\sigma} : \nabla\mathbf{u} + T^{-1}Q, \end{aligned}$$

i.e.,

$$\begin{aligned} &n d_t \ln(T^{3/2} n^{-1}) \\ &= -T^{-1}\nabla\cdot\mathbf{q} + T^{-1}\boldsymbol{\sigma} : \nabla\mathbf{u} + T^{-1}Q. \end{aligned}$$

So we define the entropy density per particle number ("molar" entropy) to be

$$s := \ln(T^{3/2} n^{-1})$$

and the entropy per volume to be

$$S := ns = n\ln(T^{3/2} n^{-1}).$$

So in the absence of heat flux, resistive heating, and viscous stress, molar entropy is conserved along particle paths.

5 Derivation of 5-moment closure

5.1 Closure requirements. To obtain closure we will invoke the following principles:

1. Momentum and energy are conserved,
2. Entropy cannot decrease,
3. Physical laws are invariant under rotation,
4. Functions can be approximated as linear.

When we study closures for the interaction of a gas with itself we will mostly assume isotropy, ignoring the symmetry-breaking effected by the presence of other species or electric and magnetic fields, except for occasional references to gyrotropy in the presence of a strong magnetic field. The justification for this is that only the magnetic field is typically strong enough that we cannot invoke approximate linearity to decouple its effects from the self-interaction of a single species.

5.2 Entropy evolution. To find closure for the 5-moment system we will require that the entropy be nondecreasing. Specifically, we require the integral over the entire domain of the entropy per volume to be nondecreasing. The entropy per volume obeys the evolution equation

$$\delta_t S = -T^{-1}\nabla\cdot\mathbf{q} + T^{-1}\boldsymbol{\sigma} : \nabla\mathbf{u} + T^{-1}Q.$$

We integrate over all space. Using the Reynolds transport theorem ($\int \delta_t = d_t \int$),

$$\begin{aligned} d_t \int S = &-\int T^{-1}\nabla\cdot\mathbf{q} \\ &+ \int T^{-1}\boldsymbol{\sigma} : \mathrm{Sym}(\nabla\mathbf{u}) + \int T^{-1}Q, \end{aligned}$$

where we have used that $\boldsymbol{\sigma} : \nabla\mathbf{u} = \boldsymbol{\sigma} : \mathrm{Sym}(\nabla\mathbf{u})$ by symmetry of $\boldsymbol{\sigma}$. This equation is a form of the general formula $ds = T^{-1}\,dq$ used to define entropy. We require that the heat flux closure, viscous stress closure, and collisional closures each independently increase the entropy.

5.3 Heat flux closure. Using integration by parts (and assuming a closed system, i.e. vanishing heat flux at the boundary),

$$-\int T^{-1}\nabla\cdot\mathbf{q} = \int q\cdot\nabla T^{-1}.$$

We are thus lead to seek a heat flux closure for which $\mathbf{q}\cdot\nabla T^{-1}$ is nonnegative. We posit that $\mathbf{q}$ is a linear function of ∇T^{-1}.

5.3.1 Istropic closure. The simplest closure assumes that this relation is isotropic, i.e.,

$$\boxed{\mathbf{q} = \overline{\kappa}\nabla T^{-1} = -\frac{\overline{\kappa}}{T^2}\nabla T},$$

where $\kappa := \frac{\overline{\kappa}}{T^2}$ is called the heat conductivity and may be determined experimentally or by a collision integral. Nonnegativity of $\mathbf{q}\cdot\nabla T^{-1} = \overline{\kappa}\|\nabla T^{-1}\|^2$ is ensured as long as $\overline{\kappa}$ is nonnegative.

5.3.2 Gyrotropic closure. In the case of nonnegligible magnetic field $\mathbf{B} = \|\mathbf{B}\|\mathbf{b}$ we instead merely assume a gyrotopic closure,

$$\mathbf{q} = \big(\kappa_\perp \mathbb{I}_\perp + \kappa_\wedge \mathbb{I}_\wedge + \kappa_\| \mathbb{I}_\|\big)\cdot\nabla T^{-1},$$

where we have used that every gyrotropic second-order tensor is a linear combination of the perpendicular, skew, and parallel gyrotropic tensors

$$\mathbb{I}_\perp := \mathbb{I} - \mathbf{bb}, \qquad \mathbb{I}_\wedge := \mathbb{I}\times\mathbf{b}, \qquad \mathbb{I}_\| := \mathbf{bb}.$$

For this closure, to ensure that $\mathbf{q}\cdot\nabla T^{-1} \geq 0$, the parallel and perpendicular heat conductivities must be nonnegative,

$$\kappa_\perp \geq 0, \qquad\qquad \kappa_\| \geq 0.$$

5.4 Viscous stress closure. To ensure that the contribution of viscous stress to entropy change is always positive, we will require that $\boldsymbol{\sigma} : \mathrm{Sym}(\nabla\mathbf{u})$ (which is the local rate of production of thermal energy) be everywhere positive. We will require that $\boldsymbol{\sigma}$ (the viscous stress) be a linear function of $\mathrm{Sym}(\nabla\mathbf{u})$ (i.e. of the gradient of velocity).

5.4.1 Isotropic closure. For a single species in the absence of a magnetic field we may assume isotropy. As argued in the appendix, if A and B are symmetric second-order tensors and A is an isotropic linear function of B, then $A = 2\mu B + \lambda\,\mathrm{tr}\,B\,\mathbb{I}$. So we may write

$$\boxed{\boldsymbol{\sigma} = \lambda\nabla\cdot\mathbf{u}\,\mathbb{I} + 2\mu\,\mathrm{Sym}(\nabla\mathbf{u})},$$

where λ and μ are called *Lamé coefficients.* Let $\underline{\underline{u}} = \mathrm{Sym}(\nabla\mathbf{u})$. Then

$$\boldsymbol{\sigma} : \nabla\mathbf{u} = \boldsymbol{\sigma}\cdot\underline{\underline{u}} = \lambda\, u_{ii}u_{jj} + 2\mu\, u_{jk}u_{jk}.$$

This needs to be positive for any $\underline{\underline{u}}$. While the coefficients of λ and μ are both nonnegative, they cannot independently be zero, so we cannot conclude that λ and μ must be positive. So we rewrite $\underline{\underline{u}}$ as the sum of a traceless and an isotropic tensor:

$$\underline{\underline{u}} = \underline{\underline{s}} + \mathbb{I}\ \mathrm{tr}\ \underline{\underline{u}}/3,$$

where $\underline{\underline{s}}$ (the rate of pure shear) is evidently $\underline{\underline{u}} - \mathbb{I}\ \mathrm{tr}\ \underline{\underline{u}}/3$, and $\mathbb{I}\ \mathrm{tr}\ \underline{\underline{u}}/3$ is the rate of hydrostatic compression. Then

$$\boldsymbol{\sigma} : \nabla\mathbf{u} = 2\mu\, s_{jk}s_{jk} + (\lambda + 2\mu/3)\, u_{ii}u_{jj}.$$

In case $\underline{\underline{u}}$ is proportional to $\mathbb{I}$, $\underline{\underline{s}}$ must be zero, and for incompressible flow u_{ii} must be zero. So the necessary (and clearly sufficient) condition for $\boldsymbol{\sigma} : \nabla\mathbf{u}$ to be nonnegative is that the viscosity μ and the "bulk modulus" $K := (\lambda + 2\mu/3)$ be nonnegative. That is,

$$\mu \geq 0, \qquad\qquad \lambda \geq -2\mu/3.$$

The Stokes assumption is that the viscous stress tensor is traceless, i.e., $\mathrm{tr}\ \boldsymbol{\sigma} = (3\lambda + 2\mu)\nabla\cdot\mathbf{u} = 0$, i.e., $K = 0$, i.e., $\lambda = -2\mu/3$.

5.4.2 Gyrotropic closure. In the presence of a magnetic field strong enough to substantially distort the particle velocity distribution from a Maxwellian we would merely require a gyrotropic linear relationship, involving as many as 12 independent parameters to be determined by experiment or collision integral.

5.5 Collisional closure. We will assume that (intraspecies) collisions involve no exchange of energy with nontranslational modes ($2\widetilde{Q} = \mathrm{tr}\ \mathbb{R} = 0$).

For interspecies collisional closure we require that

1. the total momentum be conserved,
$$\overleftrightarrow{\mathbf{R}}_{\mathrm{ie}} + \overleftrightarrow{\mathbf{R}}_{\mathrm{ei}} = 0,$$
2. the total energy be conserved,
$$\mathbf{u}_{\mathrm{i}}\cdot\overleftrightarrow{\mathbf{R}}_{\mathrm{ie}} + \overleftrightarrow{Q}_{\mathrm{ie}} + \mathbf{u}_{\mathrm{e}}\cdot\overleftrightarrow{\mathbf{R}}_{\mathrm{ei}} + \overleftrightarrow{Q}_{\mathrm{ei}} = 0,$$
3. the total entropy of the two species be nondecreasing,
$$T_{\mathrm{i}}^{-1}\overleftrightarrow{Q}_{\mathrm{ie}} + T_{\mathrm{e}}^{-1}\overleftrightarrow{Q}_{\mathrm{ei}} \geq 0.$$

We also require that the closure be a reference-frame-invariant function of the state variables (i.e. the moments) of the two gases. Thus it should be a function of the densities of the gases, the interspecies drift velocity $\mathbf{u}_\mathrm{i} - \mathbf{u}_\mathrm{e}$, their temperatures, and the magnetic field. First-order Taylor series expansion will allow us to linearize and thus decouple these dependencies.

5.5.1 Interspecies frictional drag. In case the interspecies drift velocity is zero, the resistive force $\overleftrightarrow{\mathbf{R}}_\mathrm{ie}$ should be zero.[1] Therefore we conclude that in the linear expansion $\overleftrightarrow{\mathbf{R}}_\mathrm{ie}$ should be a linear function of the drift velocity,

$$\boxed{\overleftrightarrow{\mathbf{R}}_\mathrm{ie} = \underline{\underline{\widetilde{\eta}}} \cdot (\mathbf{u}_\mathrm{e} - \mathbf{u}_\mathrm{i}),}.$$

where $\underline{\underline{\widetilde{\eta}}}$ is a gyrotropic positive-definite tensor of drag coefficients.

For a fixed velocity distribution shape, $\overleftrightarrow{\mathbf{R}}_\mathrm{ie}$ should be jointly proportional to the densities of the interacting species. So we can write

$$\underline{\underline{\widetilde{\eta}}} = -\sigma_i \sigma_e \underline{\underline{\eta}},$$

where $\sigma_i := e n_i$ and $\sigma_e := -e n_e$ are charge densities and the resistivity tensor $\underline{\underline{\eta}}$ dynamically depends solely on the temperatures or temperature tensors (since temperature is the only information available regarding the shape of the velocity distributions). In the usual quasineutral case where $n :\approx n_e \approx n_i$,

$$\boxed{\overleftrightarrow{\mathbf{R}}_\mathrm{ie} = n e \underline{\underline{\eta}} \cdot \mathbf{J}}.$$

5.5.2 Interspecies frictional heating. The rate of change of total kinetic energy due to drag is

$$\overleftrightarrow{\mathbf{R}}_\mathrm{ie} \cdot \mathbf{u}_\mathrm{i} + \overleftrightarrow{\mathbf{R}}_\mathrm{ei} \cdot \mathbf{u}_\mathrm{e} = \overleftrightarrow{\mathbf{R}}_\mathrm{ie} \cdot (\mathbf{u}_\mathrm{i} - \mathbf{u}_\mathrm{e}).$$

On the assumption of linearity we can decompose the interspecies heatings $\overleftrightarrow{Q}_\mathrm{ie}$ into frictional heating Q^f_ie (arising from interspecies drift velocity) and thermal heat exchange $Q^t_\mathrm{ie} = -Q^t_\mathrm{ei}$ which depends on the relative temperatures and densities.

Conservation of energy requires that the frictional heating balance the loss of kinetic energy due to drag:

$$Q^f_\mathrm{ie} + Q^f_\mathrm{ei} = \overleftrightarrow{\mathbf{R}}_\mathrm{ie} \cdot (\mathbf{u}_\mathrm{e} - \mathbf{u}_\mathrm{i}).$$

To ensure that entropy is respected, resistive drag warming should be positive for both species.

Consideration of collisions indicates that resistive heating should be distributed among the species inversely as the particle masses[2]:

$$\frac{Q^f_\mathrm{ie}}{Q^f_\mathrm{ei}} = \frac{m_\mathrm{e}}{m_\mathrm{i}}.$$

5.5.3 Interspecies heat transfer. In case the interspecies drift velocity is zero, heat may be exchanged. Conservation of energy says that

$$Q^t_\mathrm{ie} + Q^t_\mathrm{ei} = 0.$$

The entropy inequality says that heat cannot flow from cold to hot. So no heat should be exchanged if the temperatures of the species are equal. So a linear expansion says that

$$Q^t_\mathrm{ie} = K(T_\mathrm{e} - T_\mathrm{i}),$$

where we expect the thermal equilibration coefficient K to be proportional to the species densities and to depend on temperature.

[1] Indeed, $\overleftrightarrow{\mathbf{R}}_\mathrm{ie}$ would need to be a gyrotropic linear function of scalars (the temperatures and densities). Reflecting the domain along the axis of the magnetic field leaves the magnetic field (a pseudovector) unchanged but reverses $\overleftrightarrow{\mathbf{R}}_\mathrm{ie}$, whereas a 180-degree rotation in a plane containing the axis of the magnetic field would reverse both vectors. Composing such a rotation with a reflection, we conclude that $\overleftrightarrow{\mathbf{R}}_\mathrm{ie} = 0$.

[2] Consider a collision between two particles. In the center-of-mass frame conservation of momentum and energy say that the magnitude of the velocity does not change and the that the angle of deflection of the two particles is the same. So in the center of mass frame the ratio of the changes in kinetic energy is the reciprocal of the ratio of the masses. In symbols:

$$\begin{aligned} &m_\mathrm{i}\|\mathbf{v}'_\mathrm{i} - \mathbf{v}_\mathrm{i}\| = m_\mathrm{e}\|\mathbf{v}'_\mathrm{e} - \mathbf{v}_\mathrm{e}\|. \\ &\frac{\Delta(m_\mathrm{i}\mathbf{v}_\mathrm{i}^2)}{\Delta(m_\mathrm{e}\mathbf{v}_\mathrm{e}^2)} = \frac{m_\mathrm{i}(\mathbf{v}'_\mathrm{i} + \mathbf{v}_\mathrm{i}) \cdot (\mathbf{v}'_\mathrm{i} - \mathbf{v}_\mathrm{i})}{m_\mathrm{e}(\mathbf{v}'_\mathrm{e} + \mathbf{v}_\mathrm{e}) \cdot (\mathbf{v}'_\mathrm{e} - \mathbf{v}_\mathrm{e})} \\ &= \frac{m_\mathrm{i}\|(\mathbf{v}'_\mathrm{i} + \mathbf{v}_\mathrm{i})/2\| \cdot \|\mathbf{v}'_\mathrm{i} - \mathbf{v}_\mathrm{i}\| \cos\theta}{m_\mathrm{e}\|(\mathbf{v}'_\mathrm{e} + \mathbf{v}_\mathrm{e})/2\| \cdot \|\mathbf{v}'_\mathrm{e} - \mathbf{v}_\mathrm{e}\| \cos\theta} = \frac{\|\mathbf{v}'_\mathrm{i} - \mathbf{v}_\mathrm{i}\|}{\|\mathbf{v}'_\mathrm{e} - \mathbf{v}_\mathrm{e}\|} = \frac{m_\mathrm{e}}{m_\mathrm{i}}. \end{aligned}$$

The total entropy production is then

$$\begin{aligned}
&T_{\mathrm{i}}^{-1}Q_{\mathrm{ie}}^{t}+T_{\mathrm{e}}^{-1}Q_{\mathrm{ei}}^{t}\\
&=(T_{\mathrm{i}}^{-1}-T_{\mathrm{e}}^{-1})Q_{\mathrm{ie}}^{t}\\
&=(T_{\mathrm{i}}^{-1}-T_{\mathrm{e}}^{-1})K\rho_{\mathrm{i}}\rho_{\mathrm{e}}(T_{\mathrm{e}}-T_{\mathrm{i}})\\
&=\frac{(T_{\mathrm{e}}-T_{\mathrm{i}})^2}{T_{\mathrm{i}}T_{\mathrm{e}}}K\rho_{\mathrm{i}}\rho_{\mathrm{e}}\\
&\geq 0.
\end{aligned}$$

6 Generalization of entropy to the 10-moment case

6.1 Principal normal decomposition To study the 10-moment model it is useful to use the principal normal decompositions

$$\begin{aligned}
\mathbb{T}&=T_1\mathbf{e}_1\mathbf{e}_1+T_2\mathbf{e}_2\mathbf{e}_2+T_3\mathbf{e}_3\mathbf{e}_3,\\
\mathbb{P}&=p_1\mathbf{e}_1\mathbf{e}_1+p_2\mathbf{e}_2\mathbf{e}_2+p_3\mathbf{e}_3\mathbf{e}_3,
\end{aligned}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are eigenvectors, T_1, T_2, T_3 are the eigenvalues of $\mathbb{T}$, and $p_i = nT_i$. Recall that the scalar temperature is one third the trace of the temperature tensor, and likewise for the pressure:

$$\begin{aligned}
T&=\frac{T_1+T_2+T_3}{3},\\
p&=\frac{p_1+p_2+p_3}{3}.
\end{aligned}$$

6.2 Heuristic derivation of entropy (10-moment). A heuristic derivation of the entropy of a 10-moment system is to regard the distribution of particle velocities in each of the three principal directions as a distinct subsystem. The sum of the entropies of these subsystems is the entropy of the system as a whole.

The entropy can be understood as a state function of an equilibrium distribution. In general the equilibrium velocity distribution is Maxwellian, but if through some mysterious constraint collisions were unable to transfer energy among principal directions of the temperature tensor the equilibrium would instead be a product of 1-dimensional Maxwellian distributions (i.e. a Gaussian distribution, which is the naturally assumed distribution for the adiabatic 10-moment model) and the entropy would be the sum of the one-dimensional entropies,

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Résumé of the 5-moment closure.

To recapitulate, an entropy-respecting isotropic linearized 5-moment multi-fluid closure is

$$\begin{aligned}
&\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\\
&\partial_t(\rho\mathbf{u})+\nabla\cdot(\rho\mathbf{u}\mathbf{u})+\nabla \mathrm{p}\\
&\qquad=\frac{q}{m}\rho(\mathbf{E}+\mathbf{u}\times\mathbf{B})+\nabla\cdot\boldsymbol{\sigma}+\mathbf{R},\\
&\partial_t\mathcal{E}+\nabla\cdot(\mathbf{u}(\mathcal{E}+p))+\nabla\cdot\mathbf{q}\\
&\qquad=\nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{u})+\mathbf{J}\cdot\mathbf{E}+\mathbf{u}\cdot\mathbf{R}+Q^f+Q^t,
\end{aligned}$$

where

$$\begin{aligned}
&{}^3\!/\!_2 p:=\mathcal{E}-\rho u^2/2,\\
&\mathbf{J}:=(q/m)\rho\mathbf{u},\\
&\boldsymbol{\sigma}=2\mu(\mathrm{Sym}(\nabla\mathbf{u})-\nabla\cdot\mathbf{u}\,\mathbb{I}/3),\\
&\overleftrightarrow{\mathbf{R}}_{\mathrm{p}}=\tilde{\eta}_{\mathrm{p}}\rho\rho_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}-\mathbf{u}),\\
&\mathbf{R}=\sum_{\mathrm{p}}\overleftrightarrow{\mathbf{R}}_{\mathrm{p}}\\
&n:=\rho/m,\\
&T:=p/n,\\
&\mathbf{q}=-\tilde{\kappa}\rho\nabla T,\\
&Q^f=\sum_{\mathrm{p}}\overleftrightarrow{\mathbf{R}}_{\mathrm{p}}\cdot(\mathbf{u}_{\mathrm{p}}-\mathbf{u})\frac{m_{\mathrm{p}}}{m_{\mathrm{p}}+m},\\
&Q^t=\sum_{\mathrm{p}}\tilde{K}_{\mathrm{p}}\rho\rho_{\mathrm{p}}(T_{\mathrm{p}}-T)
\end{aligned}$$

where p denotes other species and I have altered the coefficients to suggest typical dependency on the state; the heat conductivity is $\kappa := \tilde{\kappa}\rho$. To respect entropy we require that

$$\begin{aligned}
\mu&\geq 0, &&\text{(viscosity)}\\
\tilde{\eta}_{\mathrm{p}}&\geq 0, &&\text{(drag coefficient)}\\
\tilde{\kappa}&\geq 0, &&\text{(heat flux coefficient)}\\
\tilde{K}_{\mathrm{p}}&\geq 0. &&\text{(heat transfer coefficient)}
\end{aligned}$$

---

again yielding the formula

$$2s = \ln \frac{T_1}{n^{2/3}} + \ln \frac{T_2}{n^{2/3}} + \ln \frac{T_3}{n^{2/3}} = \ln \frac{T_1 T_2 T_3}{n^2} = \ln \frac{\det \mathbb{T}}{n^2}.$$

6.3 10-moment versus 5-moment entropy. The 10-moment formula for entropy is less than or equal to the 5-moment formula:

$$\frac{1}{2} \ln \left(\frac{\det \mathbb{T}}{n^2} \right) \leq \frac{1}{2} \ln \left(\frac{T^3}{n^2} \right),$$

with equality if and only if T is isotropic; to verify this claim, use the principal normal decomposition and the monotonicity and convexity of the logarithm (so use that the arithmetic average exceeds the geometric average).

6.4 Generation of entropy (10-moment). Regarding the principal normal decomposition of the temperature/pressure tensor as a partition into subsystems, the entropy generated is the total entropy generated in the three subsystems, yielding the generic formula

$$dS = \sum_i \frac{dQ_i}{T_i} = \mathbb{T}^{-1} : \mathbb{Q} = \operatorname{tr} \left(\mathbb{T}^{-1} \cdot \mathbb{Q} \right).$$

6.5 Derivation of entropy evolution (10-moment). We repeat the derivation of entropy generalizing from the scalar case to the 10-moment case. Recall temperature tensor evolution,

$$n d_t \mathbb{T} + 2n \operatorname{Sym}(\mathbb{T} \cdot \nabla \mathbf{u}) + \nabla \cdot \mathbb{q} = \frac{q}{m} 2n \operatorname{Sym}(\mathbb{T} \times \mathbf{B}) + \mathbb{Q}.$$

In the scalar case we multiplied by T^{-1}, so to generalize we multiply by $\mathbb{T}^{-1}$; taking the trace should then give us a generalization of entropy which agrees with the scalar definition in the isotropic case.

Jacobi's formula asserts that the differential of the determinant is the trace of the matrix product of the adjugate with the differential (where the adjugate is defined by $\operatorname{adj} \mathbb{T} := (\det \mathbb{T}) \mathbb{T}^{-1}$):

$$\begin{aligned} d_t \det \mathbb{T} &= \operatorname{tr}(\operatorname{adj} \mathbb{T} \cdot d_t \mathbb{T}) \\ &= (\det \mathbb{T}) \operatorname{tr}(\mathbb{T}^{-1} \cdot d_t \mathbb{T}), \quad \text{i.e.,} \\ d_t \ln \det \mathbb{T} &= \operatorname{tr}(\mathbb{T}^{-1} \cdot d_t \mathbb{T}). \end{aligned}$$

Thus, multiplying temperature tensor evolution by $\mathbb{T}^{-1}$ (and factoring out n) and taking the trace yields:

$$n(d_t \ln \det \mathbb{T} - 2 d_t \ln n) + \mathbb{T}^{-1} : (\nabla \cdot \mathbb{q}) = \mathbb{T}^{-1} : \mathbb{Q},$$

where we have used that

$$\operatorname{tr} \left(\mathbb{T}^{-1} \cdot \operatorname{Sym}(\mathbb{T} \cdot \nabla \mathbf{u}) \right) = \nabla \cdot \mathbf{u} = -d_t \ln n$$

and that

$$\operatorname{tr} \left(\mathbb{T}^{-1} \cdot 2 \operatorname{Sym}(\mathbb{T} \times \mathbf{B}) \right) = 0.$$

So

$$n d_t \ln(n^{-2} \det \mathbb{T}) = -\mathbb{T}^{-1} : \nabla \cdot \mathbb{q} + \mathbb{T}^{-1} : \mathbb{Q}.$$

So we define the entropy density per particle number ("molar" entropy) to be

$$s := \ln \left(n^{-2} \det \mathbb{T} \right) / 2$$

(and the entropy per volume to be $S := ns$), which agrees with the scalar case.

7 Derivation of 10-moment closure

The entropy per volume obeys the evolution equation

$$2 \delta_t S = -\mathbb{T}^{-1} : (\nabla \cdot \mathbb{q}) + \mathbb{T}^{-1} : \mathbb{Q}.$$

Integrating over the domain,

$$2 d_t \int S = \int \operatorname{Sym}(\nabla \mathbb{T}^{-1}) \vdots \mathbb{q} + \int \mathbb{T}^{-1} : \mathbb{Q},$$

where we have used integration by parts and assumed vanishing heat flux at the boundary,

$$\begin{aligned} -\int \operatorname{tr} \left(\mathbb{T}^{-1} \cdot (\nabla \cdot \mathbb{q}) \right) &= \int \left(\nabla \mathbb{T}^{-1} \right) \vdots \mathbb{q} \\ &= \int \left(\operatorname{Sym}(\nabla \mathbb{T}^{-1}) \right) \vdots \mathbb{q}. \end{aligned}$$

As in the scalar case, we require that each integrand independently increase the entropy, thus decoupling the heat flux closure and the collisional closures.

7.1 10-moment heat flux closure. We require that the integrand involving heat flux be everywhere positive. Toward this end, we posit that the heat flux is a linear function of its complement in this inner product:

$$\mathbb{q} = C \vdots \operatorname{Sym} \left(\nabla \mathbb{T}^{-1} \right),$$

where C is a sixth-order tensor of coefficients.

7.1.1 Isotropic closure. The assumption of isotropy makes it easy to ensure positivity. To see how, suppose that C is an isotropic sixth-order tensor, and let A be an arbitrary *symmetric* third-order tensor. Since C is isotropic, we can write it as a linear combination of tensor products of the identity matrix, $C = \sum_p C^p$. To ensure that $A \vdots C \vdots A$ is positive it is enough to ensure that $A \vdots C^p \vdots A$ is positive. Considering representative cases, we have

$$A_{ijk}\delta_{in}\delta_{jm}\delta_{kl}A_{lmn} = A_{ijk}A_{kji} = |A|^2 \geq 0 \quad \text{and}$$
$$A_{ijk}\delta_{ij}\delta_{kl}\delta_{mn}A_{lmn} = A_{iik}A_{kmm} = |\operatorname{tr} A|^2 \geq 0;$$

note the critical role of the symmetry of A.

Assuming that the heat flux tensor is an isotropic linear function of the symmetric part of the gradient of the inverse of the temperature tensor yields the Levermore form of the closure for the heat flux tensor. As argued in the appendix, if A and B are symmetric third-order tensors and A is a linear isotropic function of B, then $A = \mu B + \lambda \operatorname{Sym}(\mathbb{I} \operatorname{tr} B)$. So

$$\boxed{\mathbb{q} = a_0 9\, \mathbb{I} \veebar \operatorname{tr}\left(\nabla \veebar \mathbb{T}^{-1}\right) + a_1 3 \nabla \veebar \mathbb{T}^{-1}},$$

where I am using the symbol $\veebar$ to denote the symmetric tensor product (that is, the symmetrization of the tensor product):

$$A \veebar B := \operatorname{Sym}(A \otimes B) \quad (\forall A, \forall B).$$

Levermore denotes the symmetric (outer) tensor product by the symbol $\vee$, but I have used a modified symbol out of concern that $\vee$ ought to be defined analogously to the wedge product.

McDonald and Groth, however, using a Chapman-Enskog expansion, obtain the following closure [McDonaldGroth08]:

$$\mathbb{q} \propto 3 \operatorname{Sym}\left(\mathbb{P} \cdot \nabla \mathbb{T}\right).$$

7.1.2 Agreement of isotropic 10-moment heat flux closure with isotropic 5-moment heat flux closure. In the isotropic case, $\mathbb{T} = T\mathbb{I}$, the trace of this closure should agree with the 5-moment heat flux closure. Assuming isotropy, $\operatorname{tr}(3\nabla \veebar \mathbb{T}^{-1}) = \operatorname{tr}(3\mathbb{I} \veebar \nabla T^{-1}) = 5\nabla T^{-1}$ and $9 \operatorname{tr}\left(\mathbb{I} \veebar \operatorname{tr}\left(\nabla \veebar \mathbb{T}^{-1}\right)\right) = 3 \operatorname{tr}\left(\mathbb{I} \veebar 5\nabla T^{-1}\right) = 25 \nabla T^{-1}$, so the trace of the 10-moment heat flux closure becomes

$$2\mathbf{q} = (a_0 25 + a_1 5)\nabla T^{-1},$$

7.1.3 Gyrotropic closure. In the case of a magnetized plasma, one would merely require that $\mathbf{q}$ be a linear *gyrotropic* function of $\operatorname{Sym}(\nabla \mathbb{T}^{-1})$, resulting in up to 26 independent parameters to be determined by experiment or collision integral.

7.2 10-moment collisional closure. For the collisional heating terms we need that $\mathbb{T}^{-1} : \mathbb{Q} = \mathbb{T}^{-1} \cdot \mathbb{R} + \sum_p \mathbb{T}^{-1} \cdot \overleftrightarrow{\mathbb{Q}}_p \geq 0$. We seek closures that ensure that each of these terms is nonnegative.

7.3 Intraspecies collisional closure. We need that $\mathbb{T}^{-1} \cdot \mathbb{R} \geq 0$. We therefore suppose that $\mathbb{R}$ is a linear function of $\mathbb{T}$. In order to invoke rotational symmetries, we will assume that this closure is independent of interaction with other species.

7.3.1 Isotropic case. In the absence of a strong magnetic field we can assume that the relation is isotropic. Then

$$\mathbb{R} = \mu \mathbb{T} + \lambda\, \mathbb{I} \operatorname{tr} \mathbb{T},$$

which we can rewrite as

$$\boxed{\mathbb{R} = \frac{\widetilde{p}\,\mathbb{I} - \mathbb{P}}{\tau},}$$

which evidently effects relaxation toward an isotropic pressure $\widetilde{p}$ with time period τ.

Assuming that intraspecies collisions conserve translational energy (i.e. involve no exchange of energy with non-translational modes), $\operatorname{tr} \mathbb{R} = 0$, i.e., $\widetilde{p} = p$. In case energy is exchanged with non-translational modes, to respect entropy $\widetilde{p}$ is constrained by the requirement that heat must not flow from cold to hot.

Henceforth assume conservation of translational thermal energy:

$$\widetilde{p} = p.$$

The rate of entropy production due to isotropization (which must be positive) is

$$\begin{aligned}\mathbb{R} : \mathbb{T}^{-1} &= \tau^{-1} n (T\,\mathbb{I} - \mathbb{T}) : \mathbb{T}^{-1} \\ &= \tau^{-1} n (\operatorname{tr} \mathbb{T} \operatorname{tr} \mathbb{T}^{-1}/3 - 3),\end{aligned}$$

I claim that this is nonnegative as long as τ is positive and is zero precisely when$\mathbb{T}$ is isotropic. That is, I claim that tr $\mathbb{T}$ tr $\mathbb{T}^{-1}/3 \geq 3$, with equality in case $\mathbb{T}$ is isotropic.

To verify this claim, use a principal normal decomposition of $\mathbb{T}_{ij}$ with eigenvalues T_1, T_2, T_3. We employ the Cauchy inequality to obtain tr $\mathbb{T}$ tr $\mathbb{T}^{-1}/9 = \frac{T_1+T_2+T_3}{3}\frac{T_1^{-1}+T_2^{-1}+T_3^{-1}}{3} = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 \geq (\mathbf{a}\cdot\mathbf{b})^2 = 1$, where we define the tuples $\mathbf{a} := \left(\sqrt{T_1}, \sqrt{T_2}, \sqrt{T_3}\right)/\sqrt{3}$ and $\mathbf{b} := \left(\sqrt{T_1^{-1}}, \sqrt{T_2^{-1}}, \sqrt{T_3^{-1}}\right)/\sqrt{3}$. (Alternatively use that the arithmetic mean exceeds the geometric mean, i.e., use convexity of the logarithm.)

7.3.2 Gyrotropic case. In the presence of a magnetic field strong enough to substantially distort the particle velocity distribution we would merely require a gyrotropic linear relationship, involving as many as 12 independent parameters to be determined by experiment or collision integral.

7.3.3 Relation of viscosity in the 5-moment closure to the time scale of relaxation to isotropy in the 10-moment closure. Just as we showed that the 10-moment closure for the heat flux tensor is consistent with the 5-moment closure for the heat flux, we now show how the 10-moment closure for collisional isotropization corresponds to the 5-moment closure for viscous stress.

To demonstrate correspondence we attempt to match up the stress tensor closure with the pressure evolution equation. We will need to suppose rapid relaxation toward isotropy, which, as we will see, means small viscosity. Observe that $\mathbb{P} = p\,\mathbb{I} - \tau\mathbb{R}$, so $\tau\mathbb{R} = \boldsymbol{\sigma}$. Pressure tensor evolution says

$$\mathbb{R} = (\delta_t p)\mathbb{I} + 2p\,\mathrm{Sym}(\nabla\mathbf{u}) + \mathcal{O}(\tau).$$

Stress closure says

$$\frac{\boldsymbol{\sigma}}{\mu} = 2\,\mathrm{Sym}(\nabla\mathbf{u}) + \frac{\lambda}{\mu}\nabla\cdot\mathbf{u}\,\mathbb{I}.$$

Rewriting pressure tensor evolution,

$$\frac{\boldsymbol{\sigma}}{p\tau} = 2\,\mathrm{Sym}(\nabla\mathbf{u}) + d_t \ln T\,\mathbb{I} + \mathcal{O}(\tau).$$

Evidently we need

$$\boxed{\mu = p\tau}$$

and $d_t \ln T = \frac{\lambda}{\mu}\nabla\cdot\mathbf{u} + \mathcal{O}(\tau)$, i.e. $d_t \ln T + \frac{\lambda}{\mu} d_t \ln n = \mathcal{O}(\tau)$, which holds (by conservation of entropy to leading order) if $\lambda/\mu = -2/3$, i.e., if the Stokes assumption that the viscous stress tensor is traceless holds. We conclude that *viscosity is equivalent to isotropization on time scales much longer than the isotropization period.*

Remark: Comparing the 10-moment isotropic intraspecies collisional closure for thermal energy with the 5-moment isotropic viscous stress closure, it is evident that the Stokes assumption is equivalent to the assumption that instraspecies collisions exchange no energy with non-translational modes.

Comparison of entropy production in 5-moment and 10-moment pressure closures. Recall that in the 5-moment model the rate of entropy production due to viscosity is $T^{-1}\boldsymbol{\sigma} : \mathrm{Sym}(\nabla\mathbf{u})$. In the 10-moment model the rate of entropy production due to isotropization is $\mathbb{T}^{-1} : \mathbb{R}$. *How do I show that these rates agree?*

7.4 Interspecies collisional closure. We seek an isotropic 10-moment collisional closure half of whose trace agrees with the isotropic 5-moment closure. Such a 10-moment closure is

$$\begin{aligned}
&\mathbf{w}_\mathrm{p} := \mathbf{u}_\mathrm{p} - \mathbf{u},\\
&\overleftrightarrow{\mathbf{R}}_\mathrm{p} = \tilde{\eta}_\mathrm{p}\rho\rho_\mathrm{p}\mathbf{w}_\mathrm{p},\\
&\overleftrightarrow{\mathbb{Q}}^f = \sum_\mathrm{p} \overleftrightarrow{\mathbb{Q}}^f_\mathrm{p},\\
&\overleftrightarrow{\mathbb{Q}}^f_\mathrm{p} = \frac{2m_\mathrm{p}}{m_\mathrm{p}+m}\overleftrightarrow{\mathbf{R}}_\mathrm{p}\cdot\left((\alpha^\parallel_\mathrm{p} - \alpha^\perp_\mathrm{p})\,\mathbb{I}\,\mathbf{w}_\mathrm{p} + \alpha^\perp_\mathrm{p}\mathbf{w}_\mathrm{p}\,\mathbb{I}\right),\\
&\overleftrightarrow{\mathbb{Q}}^t = \sum_\mathrm{p} \overleftrightarrow{\mathbb{Q}}^t_\mathrm{p},\\
&\overleftrightarrow{\mathbb{Q}}^t_\mathrm{p} = \frac{2}{3}\rho\tilde{K}_\mathrm{p}\rho_\mathrm{p}\left(\alpha^t_\mathrm{p}(\mathbb{T}_\mathrm{p} - \mathbb{T}) + \overline{\alpha}^t_\mathrm{p}\,\mathbb{I}\,(T_\mathrm{p} - T)\right),
\end{aligned}$$

where

$$\begin{aligned}
&\alpha^t_\mathrm{p} + \overline{\alpha}^t_\mathrm{p} = 1,\\
&\alpha^\parallel_\mathrm{p} + 2\alpha^\perp_\mathrm{p} = 1,
\end{aligned}$$

and where we have assumed the decomposition

$$\overleftrightarrow{\mathbb{Q}}_\mathrm{p} = \overleftrightarrow{\mathbb{Q}}^f_\mathrm{p} + \overleftrightarrow{\mathbb{Q}}^t_\mathrm{p}$$

of generalized interspecies collisional heating into a frictional drag component $\overleftrightarrow{\mathbb{Q}}^f_\mathrm{p}$ and a thermal exchange component $\overleftrightarrow{\mathbb{Q}}^t_\mathrm{p}$ and we have required that tr $\overleftrightarrow{\mathbb{Q}}^f_\mathrm{p} = 2Q^f_\mathrm{p}$ and tr $\overleftrightarrow{\mathbb{Q}}^t_\mathrm{p} = 2Q^t_\mathrm{p}$.

We now impose the requirement that entropy increase.

7.4.1 Frictional heating closure. Friction should increase the entropy for each species:

$$\mathbb{T}^{-1} : \overleftrightarrow{\mathbb{Q}}^{f}_{\mathrm{p}} \geq 0.$$

For this, making the definition $\hat{\mathbf{w}} := \mathbf{w}_{\mathrm{p}}/\|\mathbf{w}_{\mathrm{p}}\|$, we need

$$\mathbb{T}^{-1} : \Big((\alpha^{\|}_{\mathrm{p}} - \alpha^{\perp}_{\mathrm{p}})\hat{\mathbf{w}}\hat{\mathbf{w}} + \alpha^{\perp}_{\mathrm{p}}\,\mathbb{I}\Big) \geq 0,$$

i.e.,

$$\mathbb{T}^{-1} : \Big(\alpha^{\|}_{\mathrm{p}}\hat{\mathbf{w}}\hat{\mathbf{w}} + \alpha^{\perp}_{\mathrm{p}}(\mathbb{I} - \hat{\mathbf{w}}\hat{\mathbf{w}})\Big) \geq 0,$$

which, using that $\mathbb{T}$ is positive definite and that $\alpha^{\|}_{\mathrm{p}} + 2\alpha^{\perp}_{\mathrm{p}} = 1$, holds if (and, assuming that $\alpha^{\|}_{\mathrm{p}}$ and $\alpha^{\perp}_{\mathrm{p}}$ are independent of $\hat{\mathbf{w}}$, only if)

$$0 \leq 2\alpha^{\perp}_{\mathrm{p}} \leq 1.$$

That is, the portions of resistive heating in the directions parallel and perpendicular to the interspecies drift velocity must both be positive.

[This argument must be invalid, because it implies that friction should be independent of temperature. I need to compute a proper Fokker-Plank collision integral.] I contend that resistive heating should be allocated almost entirely in the directions perpendicular to the interspecies drift velocity. A basis for the assumption that we can decompose interspecies collisional heating into resistive heating and thermal equilibration is that we can perform a time-splitting where we move each species with its bulk velocity (allowing collisions to happen) and then move particles with their thermal speeds (allowing collisions to happen). In the stage where we move each species with its bulk velocity we have cold plasma and the ratio of the allocation of heating in the perpendicular direction to the allocation of heating in the parallel direction should be on the order of twice the Coulomb logarithm (typically between 10 and 20 for laboratory plasmas).

Therefore, I am inclined to choose a value of $2\alpha^{\perp}_{\mathrm{p}}$ close to 1, which says that most of the resistive heating goes into the perpendicular modes and almost none into the parallel modes. (Miura and Groth, however, choose $2\alpha^{\perp}_{\mathrm{p}} = 1/3$, so that one third of the resistive heating goes into the perpendicular directions and two thirds goes into the parallel direction.)

I remark that estimating frictional effects by regarding the thermal velocity as dominated by interspecies drift relies heavily on the validity of the time-splitting argument. Even if interspecies drift velocity initially dominates thermal velocities, this will quickly become false because resistive slowing will convert most of the kinetic energy of interspecies drift into thermal energy before half the slowing has occurred.

7.4.2 Thermal heat exchange closure. For thermal heat exchange we need the total entropy of the two species to be nondecreasing,

$$0 \leq \mathbb{T}^{-1}_{\mathrm{i}} : \overleftrightarrow{\mathbb{Q}}^{t}_{\mathrm{ie}} + \mathbb{T}^{-1}_{\mathrm{e}} : \overleftrightarrow{\mathbb{Q}}^{t}_{\mathrm{ei}}.$$

If we assume that α^{t}_{p} and $\overline{\alpha}^{t}_{\mathrm{p}}$ are fixed constants independent of species, then for this we need that

$$0 \leq \left(\mathbb{T}^{-1}_{\mathrm{i}} - \mathbb{T}^{-1}_{\mathrm{e}}\right) : \left[\alpha^{t}_{\mathrm{p}}(\mathbb{T}_{\mathrm{e}} - \mathbb{T}_{\mathrm{i}}) + \overline{\alpha}^{t}_{\mathrm{p}}\,\mathbb{I}\,(T_{\mathrm{e}} - T_{\mathrm{i}})\right].$$

Considering a case where $\mathbb{T}_{\mathrm{e}}$ and $\mathbb{T}_{\mathrm{i}}$ share common eigenvectors shows that we need $\alpha^{t}_{\mathrm{p}} = 1$ to respect positivity. Otherwise heat transfer from a highly anisotropic species can not only violate entropy but even result in a non-positive-definite pressure tensor. (Miura and Groth, however, set $\alpha^{t}_{\mathrm{p}} = 0$ and $\overline{\alpha}^{t}_{\mathrm{p}} = 1$.) A more complex entropy-respecting closure would result if we drop the dubious assumption that α^{t}_{p} and $\overline{\alpha}^{t}_{\mathrm{p}}$ are fixed constants and assume instead that collisions constantly exchange heat in both directions and that heat exchange transfers heat *away* from a temperature eigenvector of a species at a rate that increases with its energy and *into* a species at a rate that is equal among all directions.

So we need the inequality

$$0 \leq \left(\mathbb{T}^{-1}_{\mathrm{i}} - \mathbb{T}^{-1}_{\mathrm{e}}\right) : (\mathbb{T}_{\mathrm{e}} - \mathbb{T}_{\mathrm{i}}).$$

One can easily verify that this holds in case the temperature tensors (1) share common eigenvectors or (2) are nearly isotropic. Therefore, entropy is respected in the near-Maxwellian limit. *Does this inequality holds in general in case the eigenvectors are not aligned?*

8 Scaling of closure parameters

[I have only begun to write this section. I really need to work out the Fokker-Plank collision integrals.]

Résumé of the 10-moment closure.

To recapitulate, an entropy-respecting isotropic linearized 10-moment multi-fluid closure is

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbb{E} = \frac{q}{m}\rho(\mathbf{E}+\mathbf{u}\times\mathbf{B}) + \mathbf{R},\\
&\partial_t\mathbb{E} + \nabla\cdot(\mathbf{u}\mathbb{E}) + 2\,\mathrm{Sym}\,\nabla\cdot(\mathbb{P}\mathbf{u}) + \nabla\cdot\mathbb{q}\\
&\quad = \frac{q}{m}2\,\mathrm{Sym}(\rho\mathbf{u}\mathbf{E} + \mathbb{E}\times\mathbf{B})\\
&\qquad + 2\,\mathrm{Sym}(\mathbf{u}\mathbf{R}) + \mathbb{R} + \overleftrightarrow{\mathbb{Q}}^f + \overleftrightarrow{\mathbb{Q}}^t,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P} := \mathbb{E} - \rho\mathbf{u}\mathbf{u},\ \mathbb{T} := \mathbb{P}/n,\ n := \rho/m,\\
&p := \mathrm{tr}\,\mathbb{P}/3,\\
&\mathbf{w}_\mathrm{p} := \mathbf{u}_\mathrm{p} - \mathbf{u},\\
&\overleftrightarrow{\mathbf{R}}_\mathrm{p} = \tilde{\eta}_\mathrm{p}\rho\rho_\mathrm{p}\mathbf{w}_\mathrm{p},\\
&\mathbf{R} = \sum_\mathrm{p}\overleftrightarrow{\mathbf{R}}_\mathrm{p}\\
&\mathbb{q} = a_0 3\,\mathbb{I} \veebar \mathrm{tr}\left(3\nabla \veebar \mathbb{T}^{-1}\right) + a_1 3\nabla \veebar \mathbb{T}^{-1},\\
&\mathbb{R} = \frac{p\,\mathbb{I} - \mathbb{P}}{\tau},\\
&\overleftrightarrow{\mathbb{Q}}^f = \sum_\mathrm{p}\frac{2m_\mathrm{p}}{m_\mathrm{p}+m}\overleftrightarrow{\mathbf{R}}_\mathrm{p}\cdot\left((\alpha_\mathrm{p}^{\|}-\alpha_\mathrm{p}^{\perp})\,\mathbb{I}\,\mathbf{w}_\mathrm{p} + \alpha_\mathrm{p}^{\perp}\mathbf{w}_\mathrm{p}\,\mathbb{I}\right),\\
&\overleftrightarrow{\mathbb{Q}}^t = \frac{2}{3}\rho\sum_\mathrm{p}\tilde{K}_\mathrm{p}\rho_\mathrm{p}\left(\mathbb{T}_\mathrm{p}-\mathbb{T}\right),
\end{aligned}$$

where

$$\alpha_\mathrm{p}^{\|} + 2\alpha_\mathrm{p}^{\perp} = 1;$$

again, p denotes other species and I have altered the coefficients to suggest typical dependency on the state. To respect entropy we require that

$$\begin{aligned}
\tau &\geq 0, && \text{(isotropization period)}\\
\tilde{\eta}_\mathrm{p} &\geq 0, && \text{(drag coefficient)}\\
a_0, a_1 &\geq 0, && \text{(heat flux coefficients)}\\
\tilde{K}_\mathrm{p} &\geq 0, && \text{(heat transfer coefficient)}\\
1 \geq 2\alpha_\mathrm{p}^{\perp} &\geq 0. && \text{(frictional heating allocation)}
\end{aligned}$$

These coefficients are related to the coefficients of the 5-moment closure by

$$\begin{aligned}
&\mu = p\tau, && \text{(viscosity)}\\
&\kappa = \frac{25a_0 + 5a_1}{2T^2}. && \text{(heat conductivity)}
\end{aligned}$$

The basic parameter in terms of which we seek to express all other parameters is the collision period, the expected time between collisions for a given species in a given physical location. Since charged particles are constantly interacting with one another, one usually defines a charged particle to have experienced a collision when it has accumulated a net deflection of some threshold angle, say 35 degrees.

The self-collision period is the expected time required to accumulate the net threshold deflection when only deflections due to self-collision are counted. The electron-ion collision period is the expected time required to accumulate the net threshold deflection when only interspecies collisions are counted. A collision frequency is defined to be the reciprocal of its associated collision period.

Recall that the temperature is (defined to be) twice the average energy per degree of freedom. The average translational kinetic energy of a particle measured in the reference frame of the fluid is thus three halves the temperature.

We define a **thermal particle** to be a particle whose kinetic energy in the reference frame of the fluid is average, i.e., whose thermal velocity is

$$v_0 := \sqrt{\frac{T}{m}}.$$

So in thermal equilibrium typical particle velocity is inversely proportional to the square root of particle mass.

A convenient tool to estimate collision time is the collisional cross-section, which is the cross-sectional area of the sphere around a particle which represents the distance at which the Coulomb potential energy of a test particle equals the kinetic energy of a thermal particle.

The isotropization period of each species should be on the order of its collision period; it is convenient to define the collision period of each species to be its isotropization period.

The collision frequency should be proportional to the density of particles. It will depend on temperature.

Diffusive terms cause spatial oscillations to decay with a period that increases with wavelength. For a diffusive term the scale of the diffusion coefficient is determined by the requirement that the mean free path of a thermal particle matches the spatial scale of oscillations that decay over the time scale of a collision period.

For the heat flux, thermal evolution of a stationary constant-density fluid is

$$^3\!/_2 \partial_t T = \tilde{\kappa} m \nabla \cdot \nabla T.$$

Nondimensionalizing time with typical time scale t_0 and space with typical spatial scale $x_0 = v_0 t_0$, where $v_0 \sim \sqrt{\frac{T}{m}}$ is the thermal velocity, yields $\tilde{\kappa} \sim v_0^2 t_0/m = \frac{p t_0}{\rho m}$; that is,

$$\kappa \sim \frac{p t_0}{m},$$

where $\kappa := \tilde{\kappa}\rho$ is the heat conductivity.

9 Entropy of two-fluid plasma.

The entropy (per volume) of a multi-fluid plasma should be the sum of the entropies of its subsystems. In particular, the entropy of a two-fluid plasma should be the sum of the entropies of the ions, the electrons, and the electromagnetic field.

To define the entropy of the electromagnetic field we maintain the requirements that entropy should be conserved, there should be a positive-definite energy exchange between species, and the electromagnetic field should not change the entropy (because total entropy of a system can only be changed by the flow of heat).

10 Appendix

10.1 Isotropic closures. For the isotropic closures in this document we need the general form of a linear isotropic relationship between two symmetric tensors A and B of the same rank N. The coefficients of this linear relationship comprise a tensor of order $2N$.

We will use that an isotropic tensor is (up to permutation of indices) a linear combination of basis elements each of which is the tensor product of copies of the identity tensors and at most one copy of the permutation tensor. In spaces with an odd number of dimensions (namely 3) the permutation tensor has odd order, so every even-order isotropic tensor must be a linear combination of tensor products of copies of the identity tensor (see [Jeffreys72]).

The tensor of coefficients relating two order-N symmetric tensors must be symmetric in its first N and in its last N indices. Each basis element is obtained by summing (or averaging) one of the isotropic basis elements over permutations of the first indices and permutations of the last indices.

Specifically, for odd-dimensional spaces a symmetric order-N tensor A which is an isotropic linear function of a symmetric order-N tensor B is the symmetric part of a linear combination of powers of the trace operator applied to B times corresponding tensor powers of the identity tensor:

$$\begin{aligned} A = \operatorname{Sym}(&\mu_0 B + \mu_1 \mathbb{I} \operatorname{tr} B + \mu_2 \mathbb{I} \otimes \mathbb{I} \operatorname{tr} \operatorname{tr} B \\ &+ \cdots + \mu_k \mathbb{I} \otimes^k \operatorname{tr}^k B + \cdots), \end{aligned}$$

a linear combination of the integer floor of $(N+1)/2$ many basis elements.

10.1.1 Rank 1. The general isotropic linear relation between two rank-1 tensors is simply a scalar multiple:

$$A = (\lambda \mathbb{I}) \cdot B = \lambda B$$

10.1.2 Rank 2. The general (rank-2 symmetric tensor)-valued isotropic linear function A of a symmetric rank-2 tensor B is a linear combination of B and its trace times the identity matrix:

$$A = \mu B + \lambda \mathbb{I} \operatorname{tr} B$$

10.1.3 Rank 3. The general (rank-3 symmetric tensor)-valued isotropic linear function of a symmetric rank-3 tensor B is the symmetric part of a linear combination of B and its trace times the identity matrix:

$$A = \mu B + \lambda \operatorname{Sym}(\mathbb{I} \operatorname{tr} B)$$

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