Traffic flow: deriving a partial differential equation from a global conservation law,

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The objective of this note is to write a differential equation that describes the flow of traffic along a one-lane road. We adopt a continuum model of traffic flow. Rather than modeling individual cars and their velocity, we model the density of cars and their flow:

- Let $\rho(x,t)$ = density of cars (the *state variable* and
- let $f(x,t)$ = rate of flow of cars (the *flux*);

that is, $\rho(x,t)$ is the number of cars per unit length at position x and time t and $f(x,t)$ is the number of cars passing position x per unit time at time t. (Such definitions involve averaging or "smearing" cars over a region that is large relative to the size of a car but small relative to the scale that we are interested in resolving.)

Our goal is to take the statement that cars are conserved and turns it into a partial differential equation (PDE).

We begin by writing the statement that cars are conserved as a *global conservation law*. A global conservation law is expressed in terms of an arbitrary region of space $C = [x_1, x_2]$ called the *control volume* (here we will call C a *(mesh) cell*) and an arbitrary interval of time $T = [t_1, t_2]$ that we will call the *time step*. Conservation of cars says that *the number of cars in a cell at the end of the time step equals the number of cars in the cell at the beginning of the time step plus the flux of cars into the cell minus the flux of cars out of the cell.* Expressed in terms of integrals, this says:

$$\int_{x_1}^{x_2} \rho(x,t_2)\,dx = \int_{x_1}^{x_2} \rho(x,t_1)\,dx + \int_{t_1}^{t_2} f(x_1,t)\,dt - \int_{t_1}^{t_2} f(x_2,t)\,dt.$$

We can rewrite this as:

$$\int_{x_1}^{x_2} \rho(x,t_2) - \rho(x,t_1)\,dx + \int_{t_1}^{t_2} f(x_2,t) - f(x_1,t)\,dt = 0.$$

Assuming that $\rho(x,t)$ is smooth as a function of t and that $f(x,t)$ is smooth as a function of x, we can apply the fundamental theorem of calculus to rewrite this as:

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \frac{\partial \rho(x,t)}{\partial t}\,dt\,dx + \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{\partial f(x,t)}{\partial x}\,dx\,dt = 0;$$

i.e.,

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x}\,dx\,dt = 0.$$

If we assume that the integrand is piecewise continuous, this can only hold for all control volumes and control intervals if the integrand itself is zero:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x} = 0.$$