Calculation of 10-moment eigenstructure

by E. Alec Johnson, June 2008

I have verified the ten-moment eigenstructure calculated in this document with extensive computational checks.

1 The ten-moment system

1.1 Ten-moment system in conservation form

For use in shock-capturing methods, we express the ten-moment system in terms of fluxes and sources of (quasi-)conserved quantities, i.e., in the balance law form $q_t + \nabla \cdot f(q) = s(q)$. For a single species, the full 10-moment system in balance law form specifies the flux and sources of mass density ρ, momentum $\mathbf{M} = \rho\mathbf{u}$, and the energy tensor $\mathbb{E} = \rho\mathbf{uu} + \mathbb{P}$,

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{uu} + \mathbb{P}) = \frac{q}{m} \rho (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \\
&\partial_t (\rho \mathbf{uu} + \mathbb{P}) + \nabla \cdot \big(\rho \mathbf{uuu} + 3 \operatorname{Sym}(\mathbf{u}\mathbb{P})\big) = \frac{q}{m} 2 \operatorname{Sym}\big(\rho \mathbf{uE} + (\mathbb{P} + \rho \mathbf{uu}) \times \mathbf{B}\big).
\end{aligned}$$

To write this entirely in terms of conserved variables, we note that $\mathbf{u} = \mathbf{M}/\rho$ and $\mathbb{P} = \mathbb{E} - \mathbf{MM}/\rho$. So

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot \mathbf{M} = 0, \\
&\partial_t \mathbf{M} + \nabla \cdot \mathbb{E} = \frac{q}{m} (\rho \mathbf{E} + \mathbf{M} \times \mathbf{B}), \\
&\partial_t \mathbb{E} + \nabla \cdot \Big(\frac{3 \operatorname{Sym}(\mathbf{M}\mathbb{E})}{\rho} - \frac{2\mathbf{MMM}}{\rho^2} \Big) = \frac{q}{m} 2 \operatorname{Sym}\big(\mathbf{ME} + \mathbb{E} \times \mathbf{B}\big). \qquad (1.1)
\end{aligned}$$

These equations expressed for ions and electrons and combined with Maxwell's equations comprise a set of 26 equations in 26 unknowns.

1.2 Quasilinear system in primitive quantities

Shock-capturing limiters need the eigenstructure of the quasilinearized system. To calculate the eigenstructure, we put the system in quaslinear form. The eigenstructure is most easily calculated in primitive variables. In primitive variables and quasilinear form the full system is

$$\begin{aligned}
&\partial_t \rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \\
&\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla \cdot \mathbb{P}}{\rho} = \frac{q}{m} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \\
&\partial_t \mathbb{P} + \mathbf{u} \cdot \nabla \mathbb{P} + \mathbb{P} \nabla \cdot \mathbf{u} + \underline{\mathcal{PS}}(\mathbb{P} \cdot \nabla \mathbf{u}) = \frac{q}{m} \underline{\mathcal{PS}}(\mathbb{P} \times \mathbf{B})
\end{aligned}$$

1.3 Quasilinear 1-D system in primitive variables

Assuming homogeneity in all space dimensions except the first (x), the 1-D system in primitive variables becomes

$$\partial_t \rho + u_1 \partial_x \rho + \rho \partial_x u_1 = 0,$$
$$\partial_t \mathbf{u} + u_1 \partial_x \mathbf{u} + \frac{\partial_x \mathbb{P}_{1\cdot}}{\rho} = \frac{q}{m}(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$
$$\partial_t \mathbb{P} + u_1 \partial_x \mathbb{P}_{1\cdot} + \mathbb{P} \partial_x u_1 + \underline{\mathcal{PS}}(\mathbb{P}_{\cdot 1} \partial_x \mathbf{u}) = \frac{q}{m} \underline{\mathcal{PS}}(\mathbb{P} \times \mathbf{B}).$$

We align derivatives to prepare to put this quasilinear system in matrix form.

$$\begin{aligned} 0 &= \partial_t \rho + u_1 \partial_x \rho + \rho \partial_x u_1, \\ \frac{q}{m}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) &= \partial_t \mathbf{u} \qquad + u_1 \partial_x \mathbf{u} \qquad + \frac{\partial_x \mathbb{P}_{1\cdot}}{\rho}, \\ \frac{q}{m} \underline{\mathcal{PS}}(\mathbb{P} \times \mathbf{B}) &= \partial_t \mathbb{P} \qquad + \mathbb{P} \partial_x u_1 + \underline{\mathcal{PS}}(\mathbb{P}_{\cdot 1} \partial_x \mathbf{u}) + u_1 \partial_x \mathbb{P}_{1\cdot}. \end{aligned}$$

In matrix form this reads

$$\begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix}_t + \begin{bmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 1/\rho & 0 & 0 & 0 \\ 0 & 3\mathbb{P}_{11} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & 0 & 0 & 0 & 0 & 0 & u_1 \end{bmatrix} \cdot \begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix}_x = \frac{q}{m} \begin{bmatrix} 0 \\ E_1 + (u_2 B_3 - u_3 B_2) \\ E_2 + (u_3 B_1 - u_1 B_3) \\ E_3 + (u_1 B_2 - u_2 B_1) \\ 2(\mathbb{P}_{12} B_3 - \mathbb{P}_{13} B_2) \\ \mathbb{P}_{13} B_1 - \mathbb{P}_{23} B_2 + (\mathbb{P}_{22} - \mathbb{P}_{11}) B_3 \\ \mathbb{P}_{32} B_3 - \mathbb{P}_{12} B_1 + (\mathbb{P}_{11} - \mathbb{P}_{33}) B_2 \\ \mathbb{P}_{21} B_2 - \mathbb{P}_{31} B_3 + (\mathbb{P}_{33} - \mathbb{P}_{22}) B_1 \\ 2(\mathbb{P}_{23} B_1 - \mathbb{P}_{21} B_3) \\ 2(\mathbb{P}_{31} B_2 - \mathbb{P}_{32} B_1) \end{bmatrix}.$$

2 Eigenstructure for primitive variables

If we neglect the source term, the eigenvalues of the matrix represent wave speeds, and the corresponding eigenvectors represent the corresponding waves. Let $u =: u_1 + c$ denote wave speed (i.e., c is wave speed in the reference frame moving with the fluid, where $u_1 = 0$). To find the eigenstructure of the quasilinearized system we put the matrix

$$\begin{bmatrix} -c & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & 1/\rho & 0 & 0 & 0 \\ 0 & 3\mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & 0 & 0 & 0 & 0 & -c & 0 \\ 0 & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & 0 & 0 & 0 & 0 & 0 & -c \end{bmatrix}$$

in upper triangular form.

2.1 Right primitive eigenstructure

For the right eigenvectors we combine rows to do so:

$$\begin{bmatrix} -c & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho c^2 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho c^2 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho c^2 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 3\mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & 0 & 0 & 0 & 0 & -c & 0 \\ 0 & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & 0 & 0 & 0 & 0 & 0 & -c \end{bmatrix} \cdot \begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix}' = 0$$

So

$$\begin{bmatrix} -c & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\mathbb{P}_{11} - \rho c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} - \rho c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} - \rho c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & 0 & 0 & 0 & -c & 0 & 0 \\ 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & 0 & 0 & 0 & 0 & -c & 0 \\ 0 & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & 0 & 0 & 0 & 0 & 0 & -c \end{bmatrix} \cdot \begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix}' = 0$$

Denote the fast and slow speeds by

$$c_f := \sqrt{\frac{3\mathbb{P}_{11}}{\rho}}, \qquad c_s := \sqrt{\frac{\mathbb{P}_{11}}{\rho}}.$$

In the most difficult case, where $c = \pm c_f$, we have that $3\mathbb{P}_{11} - \rho c_f^2 = 0$, so $\mathbb{P}_{11} - \rho c_f^2 = -2\mathbb{P}_{11}$.

So right eigenvectors are

c:	$\pm\sqrt{\frac{3\mathbb{P}_{11}}{\rho}}$	$\pm\sqrt{\frac{\mathbb{P}_{11}}{\rho}}$	0
$\begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix}' \propto$	$\begin{bmatrix} \rho\mathbb{P}_{11} \\ \pm c_f\mathbb{P}_{11} \\ \pm c_f\mathbb{P}_{12} \\ \pm c_f\mathbb{P}_{13} \\ 3\mathbb{P}_{11}\mathbb{P}_{11} \\ 3\mathbb{P}_{12}\mathbb{P}_{11} \\ 3\mathbb{P}_{13}\mathbb{P}_{11} \\ \mathbb{P}_{23}\mathbb{P}_{11} + 2\mathbb{P}_{13}\mathbb{P}_{12} \\ \mathbb{P}_{22}\mathbb{P}_{11} + 2\mathbb{P}_{12}\mathbb{P}_{12} \\ \mathbb{P}_{33}\mathbb{P}_{11} + 2\mathbb{P}_{13}\mathbb{P}_{13} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ \pm c_s \\ 0 \\ 0 \\ \mathbb{P}_{11} \\ 0 \\ \mathbb{P}_{13} \\ 2\mathbb{P}_{12} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \pm c_s \\ 0 \\ 0 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ 0 \\ 2\mathbb{P}_{13} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2.2 Left primitive eigenstructure

To find the left eigenstructure, we combine columns to reduce to upper triangular form.

$$\begin{bmatrix}\rho\\u_1\\u_2\\u_3\\\mathbb{P}_{11}\\\mathbb{P}_{12}\\\mathbb{P}_{13}\\\mathbb{P}_{23}\\\mathbb{P}_{22}\\\mathbb{P}_{33}\end{bmatrix}^{\prime\mathrm{T}} \cdot \begin{bmatrix} -c & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -c & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -c & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -c & 0 & 0 & 1/\rho & 0 & 0 & 0\\ 0 & 3\mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0\\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0 & 0\\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} & 0 & 0 & -c & 0 & 0 & 0\\ 0 & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & 0 & 0 & 0 & -c & 0 & 0\\ 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & 0 & 0 & 0 & 0 & -c & 0\\ 0 & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & 0 & 0 & 0 & 0 & 0 & -c \end{bmatrix} = 0.$$

In case $c \neq 0$ this reduces to

$$\begin{bmatrix}\rho\\u_1\\u_2\\u_3\\\mathbb{P}_{11}\\\mathbb{P}_{12}\\\mathbb{P}_{13}\\\mathbb{P}_{23}\\\mathbb{P}_{22}\\\mathbb{P}_{33}\end{bmatrix}^{\prime\mathrm{T}} \cdot \begin{bmatrix} -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 3\mathbb{P}_{11} & 0 & 0 & 3\mathbb{P}_{11}-\rho c^2 & 0 & 0 & 0 & 0 & 0\\ 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11} & 0 & 2\mathbb{P}_{12} & \mathbb{P}_{11}-\rho c^2 & 0 & 0 & 0 & 0\\ 0 & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11} & 2\mathbb{P}_{13} & 0 & \mathbb{P}_{11}-\rho c^2 & 0 & 0 & 0\\ 0 & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & \mathbb{P}_{23} & \mathbb{P}_{31} & \mathbb{P}_{21} & -c & 0 & 0\\ 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & \mathbb{P}_{22} & 2\mathbb{P}_{12} & 0 & 0 & -c & 0\\ 0 & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & \mathbb{P}_{33} & 0 & 2\mathbb{P}_{13} & 0 & 0 & -c \end{bmatrix} = 0.$$

Here the most difficult case is $c = \pm c_s$, when $\mathbb{P}_{11} = \rho c^2$, so $3\mathbb{P}_{11} - \rho c^2 = 2\mathbb{P}_{11}$. The left eigenvectors for nonzero speeds are thus

c:	$\pm\sqrt{\frac{3\mathbb{P}_{11}}{\rho}}$	$\pm\sqrt{\frac{\mathbb{P}_{11}}{\rho}}$
$\begin{bmatrix}\rho\\u_1\\u_2\\u_3\\\mathbb{P}_{11}\\\mathbb{P}_{12}\\\mathbb{P}_{13}\\\mathbb{P}_{23}\\\mathbb{P}_{22}\\\mathbb{P}_{33}\end{bmatrix}^{\prime} \propto$	$\begin{bmatrix}0\\3\mathbb{P}_{11}\\0\\0\\\pm c_f\\0\\0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\-\mathbb{P}_{12}\mathbb{P}_{11}\\\mathbb{P}_{11}\mathbb{P}_{11}\\0\\\mp c_s\mathbb{P}_{12}\\\pm c_s\mathbb{P}_{11}\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-\mathbb{P}_{13}\mathbb{P}_{11}\\0\\\mathbb{P}_{11}\mathbb{P}_{11}\\\mp c_s\mathbb{P}_{13}\\0\\\pm c_s\mathbb{P}_{11}\\0\\0\\0\end{bmatrix}$

I don't think we really need the left eigenvectors for $c = 0$, but they are readily obtained from the original system, and here they are:

$$\begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix}' = \alpha_1 \begin{bmatrix} 3\mathbb{P}_{11} \\ 0 \\ 0 \\ 0 \\ -\rho \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4\mathbb{P}_{12}\mathbb{P}_{13} - \mathbb{P}_{23}\mathbb{P}_{11} \\ -3\mathbb{P}_{11}\mathbb{P}_{13} \\ -3\mathbb{P}_{11}\mathbb{P}_{12} \\ 3\mathbb{P}_{11}^2 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4\mathbb{P}_{12}^2 - \mathbb{P}_{11}\mathbb{P}_{22} \\ -6\mathbb{P}_{12}\mathbb{P}_{11} \\ 0 \\ 0 \\ 3\mathbb{P}_{11}^2 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4\mathbb{P}_{13}^2 - \mathbb{P}_{11}\mathbb{P}_{33} \\ 0 \\ -6\mathbb{P}_{13}\mathbb{P}_{11} \\ 0 \\ 0 \\ 3\mathbb{P}_{11}^2 \end{bmatrix}$$

3 Eigenstructure for "conserved" variables

Let

$$\underline{q} := (\rho, \mathbf{M}, \widetilde{\mathbb{E}})^{\mathrm{T}} = (\rho, \rho\mathbf{u}, \widetilde{\rho\mathbf{u}\mathbf{u}} + \widetilde{\mathbb{P}})^{\mathrm{T}}$$

denote conserved quantities, and let

$$\underline{p} := (\rho, \mathbf{u}, \widetilde{\mathbb{P}})^{\mathrm{T}} = (\rho, \mathbf{M}/\rho, \widetilde{\mathbb{E}} - \widetilde{\mathbf{M}\mathbf{M}}/\rho)^{\mathrm{T}}$$

denote primitive quantities, where for an arbitrary symmetric tensor $\mathbb{P}$ we define the tuple $\widetilde{\mathbb{P}}$ to be the six distinct components listed in the following order:

$$\widetilde{\mathbb{P}} := \left[\mathbb{P}_{11}, \mathbb{P}_{12}, \mathbb{P}_{13}, \mathbb{P}_{23}, \mathbb{P}_{22}, \mathbb{P}_{33}\right]^{\mathrm{T}}.$$

In the one-dimensional case, the balance law states

$$\underline{q}_t + f(\underline{q})_x = s(\underline{q}).$$

3.1 General theory of state variable conversion

In quasilinear form this reads

$$\underline{q}_t + f_{\underline{q}} \cdot \underline{q}_x = s.$$

Converting to primitive variables, $\underline{q}_{\underline{p}} \cdot \underline{p}_t + f_{\underline{q}} \cdot \underline{q}_{\underline{p}} \cdot \underline{p}_x = s$, i.e.,

$$\underline{p}_t + (\underline{p}_{\underline{q}} \cdot f_{\underline{q}} \cdot \underline{q}_{\underline{p}}) \cdot \underline{p}_x = s.$$

Let $\underline{q}^{\mathrm{L}}$ and $\underline{q}^{\mathrm{R}}$ denote conservative left and right eigenvectors with eigenvalue λ:

$$f_{\underline{q}} \cdot \underline{q}^{\mathrm{R}} = \lambda \underline{q}^{\mathrm{R}}, \qquad \underline{q}^{\mathrm{L}} \cdot f_{\underline{q}} = \lambda \underline{q}^{\mathrm{L}}.$$

Then

$$\underline{p}^{\mathrm{R}} := \underline{p}_{\underline{q}} \cdot \underline{q}^{\mathrm{R}}, \qquad \underline{p}^{\mathrm{L}} := \underline{q}^{\mathrm{L}} \cdot \underline{q}_{\underline{p}}$$

are the corresponding primitve left and right eigenvectors of the primitive-variable wave propagation matrix $\underline{p}_{\underline{q}} \cdot \underline{f}_{\underline{q}} \cdot \underline{q}_{\underline{p}}$. So we can calculate conservative eigenvectors from primitive eigenvectors using the relations

$$\underline{q}^{\mathrm{R}} = \underline{q}_{\underline{p}} \cdot \underline{p}^{\mathrm{R}}, \qquad\qquad \underline{q}^{\mathrm{L}} = \underline{p}^{\mathrm{L}} \cdot \underline{p}_{\underline{q}}.$$

Observe that inner product is preserved under transformation of state variables:

$$\underline{p}^{\mathrm{L}} \cdot \underline{p}^{\mathrm{R}} = \underline{q}^{\mathrm{L}} \cdot \underline{q}_{\underline{p}} \cdot \underline{p}_{\underline{q}} \cdot \underline{q}^{\mathrm{R}} = \underline{q}^{\mathrm{L}} \cdot \underline{q}^{\mathrm{R}}$$

3.2 Derivative of state variable transformation

To convert our primitive eigenvectors to conservative eigenvectors, we need to calculate the derivative of the conserved variables with respect to the primitive variables. The conversion matrix for right eigenvectors is

$$\underline{q}_{\underline{p}} = \partial \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \rho\widetilde{\mathbf{u}\mathbf{u}} + \widetilde{\mathbb{P}} \end{bmatrix} / \partial \begin{bmatrix} \rho \\ \mathbf{u} \\ \widetilde{\mathbb{P}} \end{bmatrix} = \begin{bmatrix} 1 & \underline{0}^{\mathrm{T}} & \widetilde{\underline{\underline{0}}}^{\mathrm{T}} \\ \mathbf{u} & \rho\mathbb{I} & \widetilde{\underline{\underline{0 0}}}^{\mathrm{T}} \\ \widetilde{\mathbf{u}\mathbf{u}} & \rho(\widetilde{\mathbf{u}\mathbf{u}})_{\mathbf{u}} & \widetilde{\underline{\underline{\mathbb{I}}}} \end{bmatrix},$$

(where $\mathbb{I}$ is the identity tensor), i.e.,

$$\underline{q}_{\underline{p}} = \partial \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho u_1 u_1 + \mathbb{P}_{11} \\ \rho u_1 u_2 + \mathbb{P}_{12} \\ \rho u_1 u_3 + \mathbb{P}_{13} \\ \rho u_2 u_3 + \mathbb{P}_{23} \\ \rho u_2 u_2 + \mathbb{P}_{22} \\ \rho u_3 u_3 + \mathbb{P}_{33} \end{bmatrix} / \partial \begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ \mathbb{P}_{11} \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{23} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_1 & 2\rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_2 & \rho u_2 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1 u_3 & \rho u_3 & 0 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2 u_3 & 0 & \rho u_3 & \rho u_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2 u_2 & 0 & 2\rho u_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3 u_3 & 0 & 0 & 2\rho u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The conversion matrix for left eigenvectors is slightly different:

$$\underline{p}_{\underline{q}} = \partial \begin{bmatrix} \rho \\ \mathbf{M}/\rho \\ \widetilde{\mathbb{E}} - \widetilde{\mathbf{M}\mathbf{M}}/\rho \end{bmatrix} / \partial \begin{bmatrix} \rho \\ \mathbf{M} \\ \widetilde{\mathbb{E}} \end{bmatrix} = \begin{bmatrix} 1 & \underline{0}^{\mathrm{T}} & \widetilde{\underline{\underline{0}}}^{\mathrm{T}} \\ -\mathbf{M}/\rho^2 & \mathbb{I}/\rho & \widetilde{\underline{\underline{0 0}}}^{\mathrm{T}} \\ \widetilde{\mathbf{M}\mathbf{M}}/\rho^2 & -(\widetilde{\mathbf{M}\mathbf{M}})_{\mathbf{M}}/\rho & \widetilde{\underline{\underline{\mathbb{I}}}} \end{bmatrix} = \begin{bmatrix} 1 & \underline{0}^{\mathrm{T}} & \widetilde{\underline{\underline{0}}}^{\mathrm{T}} \\ -\mathbf{u}/\rho & \mathbb{I}\rho & \widetilde{\underline{\underline{0 0}}}^{\mathrm{T}} \\ \widetilde{\mathbf{u}\mathbf{u}} & -(\widetilde{\mathbf{u}\mathbf{u}})_{\mathbf{u}} & \widetilde{\underline{\underline{\mathbb{I}}}} \end{bmatrix},$$

i.e.,

$$\underline{p}_{\underline{q}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_1/\rho & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_2/\rho & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_3/\rho & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_1 & -2u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_2 & -u_2 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1 u_3 & -u_3 & 0 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2 u_3 & 0 & -u_3 & -u_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2 u_2 & 0 & -2u_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3 u_3 & 0 & 0 & -2u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3.3 Eigenvectors for “conserved” variables

3.3.1 Right eigenvectors for “conserved” variables

Let P^{R} denote a matrix of primitive right eigenvectors. We can compute conservative right eigenvectors from the primitive right eigenvectors (without ever having to write down the quasilinearized conservative system) by the relationship

$$Q^{\mathrm{R}} = \underline{q_p} \cdot P^{\mathrm{R}},$$

where Q^{R} denotes a matrix of conservative right eigenvectors. So for the non-fast eigenvectors we have

$$Q^{\mathrm{R}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1u_1 & 2\rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1u_2 & \rho u_2 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1u_3 & \rho u_3 & 0 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2u_3 & 0 & \rho u_3 & \rho u_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2u_2 & 0 & 2\rho u_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3u_3 & 0 & 0 & 2\rho u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \pm c_s & 0 & 0 & 0 & 0 & 0 \\ 0 & \pm c_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{P}_{11} & 0 & 0 & 0 & 0 \\ \mathbb{P}_{13} & \mathbb{P}_{12} & 0 & 1 & 0 & 0 \\ 2\mathbb{P}_{12} & 0 & 0 & 0 & 1 & 0 \\ 0 & 2\mathbb{P}_{13} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \Bigg[\underbrace{\begin{bmatrix} 0 \\ 0 \\ \rho(\pm c_s) \\ 0 \\ 0 \\ \rho u_1(\pm c_s) + \mathbb{P}_{11} \\ 0 \\ \rho u_3(\pm c_s) + \mathbb{P}_{13} \\ 2\rho u_2(\pm c_s) + 2\mathbb{P}_{12} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho(\pm c_s) \\ 0 \\ 0 \\ \rho u_1(\pm c_s) + \mathbb{P}_{11} \\ \rho u_2(\pm c_s) + \mathbb{P}_{12} \\ 0 \\ 2\rho u_3(\pm c_s) + 2\mathbb{P}_{13} \end{bmatrix}}_{c=\pm c_s := \sqrt{\frac{\mathbb{P}_{11}}{\rho}}}, \underbrace{\begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ u_1u_1 \\ u_1u_2 \\ u_1u_3 \\ u_2u_3 \\ u_2u_2 \\ u_3u_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{c=0} \Bigg]$$

The fast right eigenvectors are more involved. So we just give names to the primitive components and multiply:

$$Q^{\mathrm{R}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1u_1 & 2\rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1u_2 & \rho u_2 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1u_3 & \rho u_3 & 0 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2u_3 & 0 & \rho u_3 & \rho u_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2u_2 & 0 & 2\rho u_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3u_3 & 0 & 0 & 2\rho u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \rho' := & \rho\mathbb{P}_{11} \\ \pm(u_1' := & c_f\mathbb{P}_{11}) \\ \pm(u_2' := & c_f\mathbb{P}_{12}) \\ \pm(u_3' := & c_f\mathbb{P}_{13}) \\ \mathbb{P}_{11}' := & 3\mathbb{P}_{11}\mathbb{P}_{11} \\ \mathbb{P}_{12}' := & 3\mathbb{P}_{12}\mathbb{P}_{11} \\ \mathbb{P}_{13}' := & 3\mathbb{P}_{13}\mathbb{P}_{11} \\ \mathbb{P}_{23}' := & \mathbb{P}_{23}\mathbb{P}_{11} + 2\mathbb{P}_{13}\mathbb{P}_{12} \\ \mathbb{P}_{22}' := & \mathbb{P}_{22}\mathbb{P}_{11} + 2\mathbb{P}_{12}\mathbb{P}_{12} \\ \mathbb{P}_{33}' := & \mathbb{P}_{33}\mathbb{P}_{11} + 2\mathbb{P}_{13}\mathbb{P}_{13} \end{bmatrix}.$$

So

$$
Q^{\mathrm{R}} = \begin{bmatrix}
0 & & +\,\rho' \\
\rho' u_1 & & \pm u_1'\rho \\
\rho' u_2 & & \pm u_2'\rho \\
\rho' u_3 & & \pm u_3'\rho \\
\rho' u_1 u_1 & \pm u_1'(2\rho u_1) & +\mathbb{P}_{11}' \\
\rho' u_1 u_2 & \pm u_1'\rho u_2 \pm u_2'\rho u_1 & +\mathbb{P}_{12}' \\
\rho' u_1 u_3 & \pm u_1'\rho u_3 \pm u_3'\rho u_1 & +\mathbb{P}_{13}' \\
\rho' u_2 u_3 & \pm u_2'\rho u_3 \pm u_3'\rho u_2 & +\mathbb{P}_{23}' \\
\rho' u_2 u_2 & \pm u_2'(2\rho u_2) & +\mathbb{P}_{22}' \\
\rho' u_3 u_3 & \pm u_3'(2\rho u_3) & +\mathbb{P}_{33}'
\end{bmatrix}
= \rho \begin{bmatrix}
0 \\
\mathbb{P}_{11} u_1 \\
\mathbb{P}_{11} u_2 \\
\mathbb{P}_{11} u_3 \\
\mathbb{P}_{11} u_1 (u_1 \pm 2c_f) \\
\mathbb{P}_{11} u_2 (u_1 \pm c_f) \pm c_f u_1 \mathbb{P}_{12} \\
\mathbb{P}_{11} u_3 (u_1 \pm c_f) \pm c_f u_1 \mathbb{P}_{13} \\
\mathbb{P}_{11} u_2 u_3 \pm c_f u_3 \mathbb{P}_{12} \pm c_f u_2 \mathbb{P}_{13} \\
\mathbb{P}_{11} u_2 u_2 \pm c_f u_2 2\mathbb{P}_{12} \\
\mathbb{P}_{11} u_3 u_3 \pm c_f u_3 2\mathbb{P}_{13}
\end{bmatrix}
+ \begin{bmatrix}
\rho \mathbb{P}_{11} \\
\pm c_f \mathbb{P}_{11} \rho \\
\pm c_f \mathbb{P}_{12} \rho \\
\pm c_f \mathbb{P}_{13} \rho \\
3\mathbb{P}_{11}\mathbb{P}_{11} \\
3\mathbb{P}_{12}\mathbb{P}_{11} \\
3\mathbb{P}_{13}\mathbb{P}_{11} \\
\mathbb{P}_{23}\mathbb{P}_{11} + 2\mathbb{P}_{13}\mathbb{P}_{12} \\
\mathbb{P}_{22}\mathbb{P}_{11} + 2\mathbb{P}_{12}\mathbb{P}_{12} \\
\mathbb{P}_{33}\mathbb{P}_{11} + 2\mathbb{P}_{13}\mathbb{P}_{13}
\end{bmatrix}.
$$

3.3.2 Left eigenvectors for conserved variables

Similarly, let P^{L} denote a matrix of primitive left eigenvectors. We can compute conservative left eigenvectors from the primitive left eigenvectors by the relationship

$$
Q^{\mathrm{L}} = P^{\mathrm{L}} \cdot \underline{p_q},
$$

where Q^{L} denotes a matrix of conservative left eigenvectors.

To avoid big expressions, we give a simple name to each nonzero matrix component before multiplying the matrices.

For the slow eigenvector pair for $\mathbb{P}_{12}$ and $\mathbb{P}_{13}$ define

$$
\begin{aligned}
u' &:= \mathbb{P}_{11}\mathbb{P}_{11} \\
\mathbb{P}' &:= c_s \mathbb{P}_{11},
\end{aligned}
$$

and define

$$
\begin{aligned}
u_{1b}' &:= -\mathbb{P}_{12}\mathbb{P}_{11}, & u_{1c}' &:= -\mathbb{P}_{13}\mathbb{P}_{11}, \\
u_{2b}' &:= u' & u_{3c}' &:= u' \\
\mathbb{P}_{11b}' &:= -c_s \mathbb{P}_{12}, & \mathbb{P}_{11c}' &:= -c_s \mathbb{P}_{13}, \\
\mathbb{P}_{12b}' &:= \mathbb{P}' & \mathbb{P}_{13c}' &:= \mathbb{P}'
\end{aligned}
$$

So in terms of these quantities the left eigenvectors are

$$
Q^{\mathrm{L}} = \left[\begin{bmatrix} 0 \\ 3\mathbb{P}_{11} \\ 0 \\ 0 \\ \pm c_f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\begin{bmatrix} 0 \\ u_{1b}' \\ u' \\ 0 \\ \pm \mathbb{P}_{11b}' \\ \pm \mathbb{P}' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ u_{1c}' \\ 0 \\ u' \\ \pm \mathbb{P}_{11c}' \\ 0 \\ \pm \mathbb{P}' \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]^{\mathrm{T}}
\cdot
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-u_1/\rho & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-u_2/\rho & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-u_3/\rho & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\
u_1 u_1 & -2u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
u_1 u_2 & -u_2 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
u_1 u_3 & -u_3 & 0 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\
u_2 u_3 & 0 & -u_3 & -u_2 & 0 & 0 & 0 & 1 & 0 & 0 \\
u_2 u_2 & 0 & -2u_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
u_3 u_3 & 0 & 0 & -2u_3 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

So

$$Q^{\mathrm{L}} = \left[\begin{bmatrix} -3\frac{\mathbb{P}_{11}}{\rho}u_1 \pm c_f u_1 u_1 \\ 3\frac{\mathbb{P}_{11}}{\rho} \mp 2c_f u_1 \\ 0 \\ 0 \\ \pm c_f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-u'_{1b}u_1 - u' u_2}{\rho} \pm u_1 u_1 \mathbb{P}'_{11b} \pm u_1 u_2 \mathbb{P}' \\ u'_{1b}/\rho \mp 2u_1 \mathbb{P}'_{11b} \mp u_2 \mathbb{P}' \\ u'/\rho \mp u_1 \mathbb{P}' \\ 0 \\ \pm \mathbb{P}'_{11b} \\ \pm \mathbb{P}' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-u'_{1c}u_1 - u' u_3}{\rho} \pm u_1 u_1 \mathbb{P}'_{11c} \pm u_1 u_3 \mathbb{P}' \\ u'_{1c}/\rho \mp 2u_1 \mathbb{P}'_{11c} \mp u_3 \mathbb{P}' \\ 0 \\ u'/\rho \mp u_1 \mathbb{P}' \\ \pm \mathbb{P}'_{11c} \\ 0 \\ \pm \mathbb{P}' \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]^{\mathrm{T}}$$

For the $c = 0$ eigenvectors define

$$\begin{aligned} \mathbb{P}'_{11e} &= 4\mathbb{P}_{12}\mathbb{P}_{13} - \mathbb{P}_{32}\mathbb{P}_{11}, \\ \mathbb{P}'_{12e} &:= -3\mathbb{P}_{11}\mathbb{P}_{13}, \\ \mathbb{P}'_{13e} &:= -3\mathbb{P}_{11}\mathbb{P}_{12}, \end{aligned}$$

and define

$$\begin{aligned} \mathbb{P}'_{11f} &:= 4\mathbb{P}_{12}^2 - \mathbb{P}_{11}\mathbb{P}_{22}, & \mathbb{P}'_{11g} &:= 4\mathbb{P}_{13}^2 - \mathbb{P}_{11}\mathbb{P}_{33}, \\ \mathbb{P}'_{12f} &:= -6\mathbb{P}_{12}\mathbb{P}_{11}, & \mathbb{P}'_{13g} &:= -6\mathbb{P}_{13}\mathbb{P}_{11}. \end{aligned}$$

The left eigenvectors for $c = 0$ are given by

$$Q^{\mathrm{L}} = \left[\begin{bmatrix} 3\mathbb{P}_{11} \\ 0 \\ 0 \\ 0 \\ -\rho \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathbb{P}'_{11e} \\ \mathbb{P}'_{12e} \\ \mathbb{P}'_{13e} \\ 3\mathbb{P}_{11}^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathbb{P}'_{11f} \\ \mathbb{P}'_{12f} \\ 0 \\ 0 \\ 3\mathbb{P}_{11}^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathbb{P}'_{11g} \\ 0 \\ \mathbb{P}'_{13g} \\ 0 \\ 0 \\ 3\mathbb{P}_{11}^2 \end{bmatrix}\right]^{\mathrm{T}} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_1/\rho & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_2/\rho & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_3/\rho & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1u_1 & -2u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1u_2 & -u_2 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1u_3 & -u_3 & 0 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2u_3 & 0 & -u_3 & -u_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2u_2 & 0 & -2u_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3u_3 & 0 & 0 & -2u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So the $c = 0$ left eigenvectors are

$$\underline{q}^{\mathrm{L}} = \begin{bmatrix} 3\mathbb{P}_{11} - \rho u_1 u_1 \\ 2\rho u_1 \\ 0 \\ 0 \\ -\rho \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} u_1 u_1 \mathbb{P}'_{11e} + u_1 u_2 \mathbb{P}'_{12e} + u_1 u_3 \mathbb{P}'_{13e} + 3u_2 u_3 \mathbb{P}^2_{11} \\ -(2u_1 \mathbb{P}'_{11e} + u_2 \mathbb{P}'_{12e} + u_3 \mathbb{P}'_{13e}) \\ -(u_1 \mathbb{P}'_{12e} + 3u_3 \mathbb{P}^2_{11}) \\ -(u_1 \mathbb{P}'_{13e} + 3u_2 \mathbb{P}^2_{11}) \\ \mathbb{P}'_{11e} \\ \mathbb{P}'_{12e} \\ \mathbb{P}'_{13e} \\ 3\mathbb{P}^2_{11} \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} u_1 u_1 \mathbb{P}'_{11f} + u_1 u_2 \mathbb{P}'_{12f} + u_2 u_2 3\mathbb{P}^2_{11} \\ -(2u_1 \mathbb{P}'_{11f} + u_2 \mathbb{P}'_{12f}) \\ -(u_1 \mathbb{P}'_{12f} + 6u_2 \mathbb{P}^2_{11}) \\ 0 \\ \mathbb{P}'_{11f} \\ \mathbb{P}'_{12f} \\ 0 \\ 0 \\ 3\mathbb{P}^2_{11} \\ 0 \end{bmatrix}, \begin{bmatrix} u_1 u_1 \mathbb{P}'_{11g} + u_1 u_3 \mathbb{P}'_{13g} + u_3 u_3 3\mathbb{P}^2_{11} \\ -(2u_1 \mathbb{P}'_{11g} + u_3 \mathbb{P}'_{13g}) \\ 0 \\ -(u_1 \mathbb{P}'_{13g} + 6u_3 \mathbb{P}^2_{11}) \\ \mathbb{P}'_{11g} \\ 0 \\ \mathbb{P}'_{13g} \\ 0 \\ 0 \\ 3\mathbb{P}^2_{11} \end{bmatrix}.$$

4 Quasilinear system in conserved variables

Recall the ten-moment system (1.1),

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \mathbf{M} = 0, \\ &\partial_t \mathbf{M} + \nabla \cdot \mathbb{E} = \frac{q}{m}(\rho \mathbf{E} + \mathbf{M} \times \mathbf{B}), \\ &\partial_t \mathbb{E} + \nabla \cdot \Big(\frac{3\,\mathrm{Sym}(\mathbf{M}\mathbb{E})}{\rho} - \frac{2\mathbf{MMM}}{\rho^2} \Big) = \frac{q}{m} 2\,\mathrm{Sym}\big(\mathbf{ME} + \mathbb{E} \times \mathbf{B}\big). \end{aligned}$$

In one dimension this reads

$$\begin{aligned} &\partial_t \rho + \partial_x \mathbf{M}_1 = 0, \\ &\partial_t \mathbf{M} + \partial_x \mathbb{E}_{1\cdot} = \frac{q}{m}(\rho \mathbf{E} + \mathbf{M} \times \mathbf{B}), \\ &\partial_t \mathbb{E} + \partial_x \Big(\frac{\mathbf{M}_1 \mathbb{E} + 2\,\mathrm{Sym}(\mathbb{E}_{1\cdot}\mathbf{M})}{\rho} - \frac{2\mathbf{M}_1\mathbf{MM}}{\rho^2} \Big) = \frac{q}{m} 2\,\mathrm{Sym}\big(\mathbf{ME} + \mathbb{E} \times \mathbf{B}\big). \end{aligned}$$

Observe that the flux is

$$f = \begin{bmatrix} \mathbf{M}_1 \\ \mathbb{E}_{1\cdot} \\ \frac{\mathbf{M}_1 \mathbb{E} + 2\,\mathrm{Sym}(\mathbb{E}_{1\cdot}\mathbf{M})}{\rho} - \frac{2\mathbf{M}_1\mathbf{MM}}{\rho^2} \end{bmatrix}$$

To quasilinearize we calculate the derivative of the x-coordinate of the flux.

$$f_{\underline{q}} = \partial \begin{bmatrix} \mathbf{M}_1 \\ \mathbb{E}_{1.} \\ \frac{\mathbf{M}_1 \mathbb{E} + 2\,\mathrm{Sym}(\mathbb{E}_1 . \mathbf{M})}{\rho} - \frac{2\mathbf{M}_1 \mathbf{M}\mathbf{M}}{\rho^2} \end{bmatrix} / \partial \begin{bmatrix} \rho \\ \mathbf{M} \\ \mathbb{E} \end{bmatrix} = \partial \begin{bmatrix} \mathbf{M}_1 \\ \mathbb{E}_{11} \\ \mathbb{E}_{12} \\ \mathbb{E}_{13} \\ \frac{3\mathbb{E}_{11}\mathbf{M}_1}{\rho} - \frac{2\mathbf{M}_1^3}{\rho^2} \\ \frac{\mathbb{E}_{11}\mathbf{M}_2 + 2\mathbb{E}_{12}\mathbf{M}_1}{\rho} - \frac{2\mathbf{M}_1^2\mathbf{M}_2}{\rho^2} \\ \frac{\mathbb{E}_{11}\mathbf{M}_3 + 2\mathbb{E}_{13}\mathbf{M}_1}{\rho} - \frac{2\mathbf{M}_1^2\mathbf{M}_3}{\rho^2} \\ \frac{\mathbf{M}_1\mathbb{E}_{23} + \mathbb{E}_{12}\mathbf{M}_3 + \mathbb{E}_{13}\mathbf{M}_2}{\rho} - \frac{2\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3}{\rho^2} \\ \frac{\mathbf{M}_1\mathbb{E}_{22} + 2\mathbb{E}_{12}\mathbf{M}_2}{\rho} - \frac{2\mathbf{M}_1\mathbf{M}_2^2}{\rho^2} \\ \frac{\mathbf{M}_1\mathbb{E}_{33} + 2\mathbb{E}_{13}\mathbf{M}_3}{\rho} - \frac{2\mathbf{M}_1\mathbf{M}_3^2}{\rho^2} \end{bmatrix} / \partial \begin{bmatrix} \rho \\ \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{M}_3 \\ \mathbb{E}_{11} \\ \mathbb{E}_{12} \\ \mathbb{E}_{13} \\ \mathbb{E}_{23} \\ \mathbb{E}_{22} \\ \mathbb{E}_{33} \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{3\mathbb{E}_{11}\mathbf{M}_1}{\rho^2} + \frac{4\mathbf{M}_1^3}{\rho^3} & \frac{3\mathbb{E}_{11}}{\rho} - \frac{6\mathbf{M}_1^2}{\rho^2} & 0 & 0 & \frac{3\mathbf{M}_1}{\rho} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\mathbb{E}_{11}\mathbf{M}_2 + 2\mathbb{E}_{12}\mathbf{M}_1}{\rho^2} + \frac{4\mathbf{M}_1^2\mathbf{M}_2}{\rho^3} & \frac{2\mathbb{E}_{12}}{\rho} - \frac{4\mathbf{M}_1\mathbf{M}_2}{\rho^2} & \frac{\mathbb{E}_{11}}{\rho} - \frac{2\mathbf{M}_1^2}{\rho^2} & 0 & \frac{\mathbf{M}_2}{\rho} & \frac{2\mathbf{M}_1}{\rho} & 0 & 0 & 0 & 0 \\ -\frac{\mathbb{E}_{11}\mathbf{M}_3 + 2\mathbb{E}_{13}\mathbf{M}_1}{\rho^2} + \frac{4\mathbf{M}_1^2\mathbf{M}_3}{\rho^3} & \frac{2\mathbb{E}_{13}}{\rho} - \frac{4\mathbf{M}_1\mathbf{M}_3}{\rho^2} & 0 & \frac{\mathbb{E}_{11}}{\rho} - \frac{2\mathbf{M}_1^2}{\rho^2} & \frac{\mathbf{M}_3}{\rho} & 0 & \frac{2\mathbf{M}_1}{\rho} & 0 & 0 & 0 \\ -\frac{\mathbf{M}_1\mathbb{E}_{23} + \mathbb{E}_{12}\mathbf{M}_3 + \mathbb{E}_{13}\mathbf{M}_2}{\rho^2} + \frac{4\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3}{\rho^3} & \frac{\mathbb{E}_{23}}{\rho} - \frac{2\mathbf{M}_2\mathbf{M}_3}{\rho^2} & \frac{\mathbb{E}_{13}}{\rho} - \frac{2\mathbf{M}_1\mathbf{M}_3}{\rho^2} & \frac{\mathbb{E}_{12}}{\rho} - \frac{2\mathbf{M}_1\mathbf{M}_2}{\rho^2} & 0 & \frac{\mathbf{M}_3}{\rho} & \frac{\mathbf{M}_2}{\rho} & \frac{\mathbf{M}_1}{\rho} & 0 & 0 \\ -\frac{\mathbf{M}_1\mathbb{E}_{22} + 2\mathbb{E}_{12}\mathbf{M}_2}{\rho^2} + \frac{4\mathbf{M}_1\mathbf{M}_2^2}{\rho^3} & \frac{\mathbb{E}_{22}}{\rho} - \frac{2\mathbf{M}_2^2}{\rho^2} & \frac{2\mathbb{E}_{12}}{\rho} - \frac{4\mathbf{M}_1\mathbf{M}_2}{\rho^2} & 0 & 0 & \frac{2\mathbf{M}_2}{\rho} & 0 & 0 & \frac{\mathbf{M}_1}{\rho} & 0 \\ -\frac{\mathbf{M}_1\mathbb{E}_{33} + 2\mathbb{E}_{13}\mathbf{M}_3}{\rho^2} + \frac{4\mathbf{M}_1\mathbf{M}_3^2}{\rho^3} & \frac{\mathbb{E}_{33}}{\rho} - \frac{2\mathbf{M}_3^2}{\rho^2} & 0 & \frac{2\mathbb{E}_{13}}{\rho} - \frac{4\mathbf{M}_1\mathbf{M}_3}{\rho^2} & 0 & 0 & \frac{2\mathbf{M}_3}{\rho} & 0 & 0 & \frac{\mathbf{M}_1}{\rho} \end{bmatrix}$$