Two-fluid Source Term ODE

by E. Alec Johnson, 2011.

1 Basic Equations

If we neglect spatial derivatives, then the two-fluid-Maxwell equations reduce to an ODE. The purpose of this note is to solve this **source term ODE**; we assume throughout that spatial derivatives are zero. Neglecting spatial derivatives, the two-fluid equations are the source terms of Maxwell's equations coupled to the source terms of the gas-dynamics equations for each species.

Maxwell's equations assert that the magnetic field is constant and that the displacement current balances the net electrical current:

$$\begin{aligned}
&\partial_t \mathbf{B} = 0,\\
&\partial_t \mathbf{E} = -\mathbf{J}/\varepsilon_0 = e(n_\mathrm{e}\mathbf{u}_\mathrm{e} - n_\mathrm{i}\mathbf{u}_i)/\varepsilon_0.
\end{aligned}$$

The density evolution equations assert that the densities (whether mass density or particle density or charge density) remain constant:

$$\partial_t\rho_\mathrm{s} = 0, \quad \text{i.e.,} \quad \partial_t n_\mathrm{s} = 0, \quad \text{i.e.,} \quad \partial_t\sigma_\mathrm{s} = 0.$$

The momentum equation for species s is

$$\partial_t(\rho_\mathrm{s}\mathbf{u}_\mathrm{s}) = \frac{q_\mathrm{s}}{m_\mathrm{s}}\rho_\mathrm{s}(\mathbf{E} + \mathbf{u}_\mathrm{s}\times\mathbf{B}) + \mathbf{R}_\mathrm{s}.$$

We will neglect the collisional drag force $\mathbf{R}_\mathrm{s}$. Since densities are constant, we can divide by density.

We thus get the electro-momentum system

$$\begin{aligned}
&\partial_t \mathbf{E} = e(n_\mathrm{e}\mathbf{u}_\mathrm{e} - n_\mathrm{i}\mathbf{u}_i)/\varepsilon_0,\\
&\partial_t \mathbf{u}_\mathrm{i} = \frac{e}{m_\mathrm{i}}(\mathbf{E} + \mathbf{u}_\mathrm{i}\times\mathbf{B}),\\
&\partial_t \mathbf{u}_\mathrm{e} = \frac{-e}{m_\mathrm{e}}(\mathbf{E} + \mathbf{u}_\mathrm{e}\times\mathbf{B}).
\end{aligned} \tag{1}$$

Evolution of energy is implied by evolution of momentum (which implies evolution of kinetic energy) and evolution of pressure (which is equivalent to evolution of thermal energy). Note that pressure evolution is temperature evolution times the constant density.

The five-moment pressure evolution equation

$$(3/2)\partial_t p_\mathrm{s} = Q_\mathrm{s}$$

says that pressure is constant in the absence of interspecies collisional heating due to resistive drag and thermal equilibration. If the drag force is non-negligible, and assuming that the resistive drag coefficient is a function of temperature, the electro-momentum system coupled to pressure evolution comprises a minimally closed system. Neglecting collisional terms, pressure evolution simply asserts that pressure is constant.

2 The electro-momentum system

Written in matrix form, the non-resistive electro-momentum system (1) reads

$$\partial_t \begin{bmatrix}\mathbf{E}\\ \mathbf{u}_\mathrm{i}\\ \mathbf{u}_\mathrm{e}\end{bmatrix} = \begin{bmatrix} 0 & -\frac{en_\mathrm{i}}{\varepsilon_0} & \frac{en_\mathrm{e}}{\varepsilon_0}\\ \frac{e}{m_i} & -\frac{e\mathbf{B}}{m_\mathrm{i}}\times\mathbb{1} & 0\\ -\frac{e}{m_e} & 0 & \frac{e\mathbf{B}}{m_\mathrm{e}}\times\mathbb{1}\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{u}_\mathrm{i}\\ \mathbf{u}_\mathrm{e}\end{bmatrix}.$$

We can make this ODE antisymmetric by rescaling. For a generic rescaling, suppose

$$\begin{aligned}
\mathbf{E} &= \widetilde{\mathbf{E}}\mathbf{E}_0,\\
\mathbf{u}_\mathrm{i} &= \widetilde{\mathbf{u}}_\mathrm{i}\mathbf{u}_{\mathrm{i}0},\\
\mathbf{u}_\mathrm{e} &= \widetilde{\mathbf{u}}_\mathrm{e}\mathbf{u}_{\mathrm{e}0}.
\end{aligned}$$

Making this substitution gives the system

$$\partial_t \begin{bmatrix}\widetilde{\mathbf{E}}\\ \widetilde{\mathbf{u}}_\mathrm{i}\\ \widetilde{\mathbf{u}}_\mathrm{e}\end{bmatrix} = \begin{bmatrix} 0 & -\frac{en_\mathrm{i}}{\varepsilon_0}\frac{\mathbf{u}_{\mathrm{i}0}}{\mathbf{E}_0} & \frac{en_\mathrm{e}}{\varepsilon_0}\frac{\mathbf{u}_{\mathrm{e}0}}{\mathbf{E}_0}\\ \frac{e}{m_i}\frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{i}0}} & -\frac{e\mathbf{B}}{m_\mathrm{i}}\times\mathbb{1} & 0\\ -\frac{e}{m_e}\frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{e}0}} & 0 & \frac{e\mathbf{B}}{m_\mathrm{e}}\times\mathbb{1}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}\\ \widetilde{\mathbf{u}}_\mathrm{i}\\ \widetilde{\mathbf{u}}_\mathrm{e}\end{bmatrix}.$$

If we require this system to be antisymmetric then

$$\frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{i}0}} = \sqrt{\frac{\rho_\mathrm{i}}{\varepsilon_0}} \quad \text{and} \quad \frac{\mathbf{E}_0}{\mathbf{u}_{\mathrm{e}0}} = \sqrt{\frac{\rho_\mathrm{e}}{\varepsilon_0}},$$

(where recall that $\rho_\mathrm{i} = m_\mathrm{i}n_\mathrm{i}$ and $\rho_\mathrm{e} = m_\mathrm{e}n_\mathrm{e}$) and the system becomes

$$\partial_t \begin{bmatrix}\widetilde{\mathbf{E}}\\ \widetilde{\mathbf{u}}_\mathrm{i}\\ \widetilde{\mathbf{u}}_\mathrm{e}\end{bmatrix} = \begin{bmatrix} 0 & -\omega_\mathrm{i}\,\mathbb{1} & \omega_\mathrm{e}\,\mathbb{1}\\ \omega_\mathrm{i}\,\mathbb{1} & -\mathbf{B}_\mathrm{i}\times\mathbb{1} & 0\\ -\omega_\mathrm{e}\,\mathbb{1} & 0 & -\mathbf{B}_\mathrm{e}\times\mathbb{1}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}\\ \widetilde{\mathbf{u}}_\mathrm{i}\\ \widetilde{\mathbf{u}}_\mathrm{e}\end{bmatrix},$$

where each entry in the block matrix represents a 3×3 matrix and where

$$\omega_\mathrm{i} = e\sqrt{\frac{n_\mathrm{i}}{\varepsilon_0 m_\mathrm{i}}} \quad \text{and} \quad \omega_\mathrm{e} = e\sqrt{\frac{n_\mathrm{e}}{\varepsilon_0 m_\mathrm{e}}}$$

denote the ion and electron plasma frequencies and

$$\mathbf{B}_\mathrm{i} = \frac{e\mathbf{B}}{m_\mathrm{i}} \quad \text{and} \quad \mathbf{B}_\mathrm{e} = \frac{-e\mathbf{B}}{m_\mathrm{e}}$$

are the magnetic field rescaled for ions and electrons. Their magnitudes are the ion gyrofrequency $\Omega_\mathrm{i} := |\mathbf{B}_\mathrm{i}|$ and the electron gyrofrequency $\Omega_\mathrm{e} := |\mathbf{B}_\mathrm{e}|$.

3 Solution of perpendicular system

To solve the system we decompose into parallel and perpendicular components. Without loss of generality assume that $\mathbf{B}$ is in the direction of the third axis. Then our system decouples into a parallel system

$$\partial_t \begin{bmatrix}\widetilde{\mathbf{E}}_3\\ \widetilde{\mathbf{u}}_{\mathrm{i}3}\\ \widetilde{\mathbf{u}}_{\mathrm{e}3}\end{bmatrix} = \begin{bmatrix} 0 & -\omega_\mathrm{i} & \omega_\mathrm{e}\\ \omega_\mathrm{i} & 0 & 0\\ -\omega_\mathrm{e} & 0 & 0\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}_3\\ \widetilde{\mathbf{u}}_{\mathrm{i}3}\\ \widetilde{\mathbf{u}}_{\mathrm{e}3}\end{bmatrix},$$

and a perpendicular system

$$\partial_t \begin{bmatrix}\widetilde{\mathbf{E}}_1\\ \widetilde{\mathbf{E}}_2\\ \widetilde{\mathbf{u}}_{\mathrm{i}1}\\ \widetilde{\mathbf{u}}_{\mathrm{i}2}\\ \widetilde{\mathbf{u}}_{\mathrm{e}1}\\ \widetilde{\mathbf{u}}_{\mathrm{e}2}\end{bmatrix} = \begin{bmatrix} 0 & 0 & -\omega_\mathrm{i} & 0 & \omega_\mathrm{e} & 0\\ 0 & 0 & 0 & -\omega_\mathrm{i} & 0 & \omega_\mathrm{e}\\ \omega_\mathrm{i} & 0 & 0 & \Omega_\mathrm{i} & 0 & 0\\ 0 & \omega_\mathrm{i} & -\Omega_\mathrm{i} & 0 & 0 & 0\\ -\omega_\mathrm{e} & 0 & 0 & 0 & 0 & -\Omega_\mathrm{e}\\ 0 & -\omega_\mathrm{e} & 0 & 0 & \Omega_\mathrm{e} & 0\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{E}}_1\\ \widetilde{\mathbf{E}}_2\\ \widetilde{\mathbf{u}}_{\mathrm{i}1}\\ \widetilde{\mathbf{u}}_{\mathrm{i}2}\\ \widetilde{\mathbf{u}}_{\mathrm{e}1}\\ \widetilde{\mathbf{u}}_{\mathrm{e}2}\end{bmatrix}.$$

This is an antisymmetric matrix and therefore has imaginary eigenvalues and orthogonal eigenvectors. If we view the first and second components of each vector as real and imaginary parts, then this becomes a 3×3 complex linear differential equation with a skew hermitian coefficient matrix:

$$\partial_t \begin{bmatrix} \widetilde{\mathbf{E}}_\perp \\ \widetilde{\mathbf{u}}_{\mathrm{i}\perp} \\ \widetilde{\mathbf{u}}_{\mathrm{e}\perp} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{\mathrm{i}} & \omega_{\mathrm{e}} \\ \omega_{\mathrm{i}} & -i\Omega_{\mathrm{i}} & 0 \\ -\omega_{\mathrm{e}} & 0 & i\Omega_{\mathrm{e}} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{E}}_\perp \\ \widetilde{\mathbf{u}}_{\mathrm{i}\perp} \\ \widetilde{\mathbf{u}}_{\mathrm{e}\perp} \end{bmatrix}, \qquad (2)$$

where we have used the natural isomorphism between $SO(2,\mathbb{R})$ and complex numbers

$$a + ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Observe that the parallel system is the special case of this system when the magnetic field is zero.

To generalize, suppose we want to solve the constant-coefficient linear ODE

$$\underline{x}' = \underline{\underline{A}} \cdot \underline{x}.$$

Seeking a solution $\underline{x}(t) = \underline{v}\exp(\lambda t)$ (where $\underline{v} \neq \underline{0}$) leads to the eigenvector problem

$$\underline{v}\lambda = \underline{\underline{A}} \cdot \underline{v}, \quad \text{i.e.,} \quad (\underline{\underline{A}} - \mathbb{1}\,\lambda)\cdot\underline{v} = 0.$$

We recall the theory of skew-Hermitian and Hermitian matrices. Since $\underline{\underline{A}}$ is skew-Hermitian (i.e. $\underline{\underline{A}}^* = -\underline{\underline{A}}$, where * denotes the conjugate of the transpose), $\underline{\underline{B}} := i\underline{\underline{A}}$ is Hermitian (i.e. $\underline{\underline{B}}^* = \underline{\underline{B}}$).

The eigenvalues of a Hermitian matrix are real. Indeed, assuming without loss of generality that $\underline{v}^*\underline{v} = 1$,

$$\begin{aligned} \lambda &= \underline{v}^*\underline{v}\lambda = \underline{v}^*\underline{\underline{B}}\underline{v} = \underline{v}^*\underline{\underline{B}}^*\underline{v} = (\underline{v}^*\underline{\underline{B}}\underline{v})^* = (\underline{v}^*\underline{v}\lambda)^* \\ &= \underline{v}^*\underline{v}\lambda^* = \lambda^*, \end{aligned}$$

and eigenvectors for different eigenvalues are orthogonal:

$$\begin{aligned} \underline{v}_2^*\underline{v}_1\lambda_1 &= \underline{v}_2^*\underline{\underline{B}}\underline{v}_1 = \underline{v}_2^*\underline{\underline{B}}^*\underline{v}_1 = (\underline{v}_1^*\underline{\underline{B}}\underline{v}_2)^* = (\underline{v}_1^*\underline{v}_2\lambda_2)^* \\ &= \underline{v}_2^*\underline{v}_1\lambda_2, \end{aligned}$$

which says that either $\underline{v}_2^*\underline{v}_1 = 0$ or $\lambda_1 = \lambda_2$.

Note that if $(\underline{v}, \omega)$ is an eigenvector-eigenvalue pair for $\underline{\underline{B}}$ then $(\underline{v}, i\omega)$ is an eigenvector-eigenvalue pair for $\underline{\underline{A}}$.

To find the eigenstructure we solve

$$0 = (\underline{\underline{A}} - i\omega)\cdot\underline{v} = \begin{bmatrix} -i\omega & -\omega_{\mathrm{i}} & \omega_{\mathrm{e}} \\ \omega_{\mathrm{i}} & -i(\Omega_{\mathrm{i}} + \omega) & 0 \\ -\omega_{\mathrm{e}} & 0 & i(\Omega_{\mathrm{e}} - \omega) \end{bmatrix} \cdot \underline{v}. \qquad (3)$$

If this has a nontrivial solution then the first row is a linear combination of the second two and we can ignore it. The second two equations then show that an eigenvector must be a multiple of the form

$$\underline{v} = \begin{bmatrix} i\beta_{\mathrm{e}}\beta_{\mathrm{i}} \\ \omega_{\mathrm{i}}\beta_{\mathrm{e}} \\ \omega_{\mathrm{e}}\beta_{\mathrm{i}} \end{bmatrix},$$

where

$$\begin{aligned} &\beta_{\mathrm{i}} := \Omega_{\mathrm{i}} + \omega \quad \text{and} \\ &\beta_{\mathrm{e}} := \Omega_{\mathrm{e}} - \omega, \end{aligned}$$

as is confirmed (for the last two rows) by computing $(\underline{\underline{A}} - i\omega)\cdot\underline{v}$; the relation implied by the first row reveals the characteristic equation. Alternatively, the calculation

$$\begin{aligned} \underline{\underline{A}}\cdot\underline{v} &= \begin{bmatrix} 0 & -\omega_{\mathrm{i}} & \omega_{\mathrm{e}} \\ \omega_{\mathrm{i}} & -i\Omega_{\mathrm{i}} & 0 \\ -\omega_{\mathrm{e}} & 0 & i\Omega_{\mathrm{e}} \end{bmatrix} \cdot \begin{bmatrix} i\beta_{\mathrm{e}}\beta_{\mathrm{i}} \\ \omega_{\mathrm{i}}\beta_{\mathrm{e}} \\ \omega_{\mathrm{e}}\beta_{\mathrm{i}} \end{bmatrix} \\ &= \begin{bmatrix} -\omega_{\mathrm{i}}^2\beta_{\mathrm{e}} + \omega_{\mathrm{e}}^2\beta_{\mathrm{i}} \\ i\beta_{\mathrm{i}}(\omega_{\mathrm{i}}\beta_{\mathrm{e}}) - i\Omega_{\mathrm{i}}(\omega_{\mathrm{i}}\beta_{\mathrm{e}}) \\ -i(\omega_{\mathrm{e}}\beta_{\mathrm{i}})\beta_{\mathrm{e}} + i\Omega_{\mathrm{e}}(\omega_{\mathrm{e}}\beta_{\mathrm{i}}) \end{bmatrix} = \underline{v}i\omega = \begin{bmatrix} i\beta_{\mathrm{e}}\beta_{\mathrm{i}} \\ \omega_{\mathrm{i}}\beta_{\mathrm{e}} \\ \omega_{\mathrm{e}}\beta_{\mathrm{i}} \end{bmatrix} i\omega \end{aligned}$$

shows that ω must satisfy

$$\begin{aligned} &\beta_{\mathrm{e}}\beta_{\mathrm{i}}\omega = \omega_{\mathrm{i}}^2\beta_{\mathrm{e}} - \omega_{\mathrm{e}}^2\beta_{\mathrm{i}}, \\ &\omega = \beta_{\mathrm{i}} - \Omega_{\mathrm{i}}, \\ &\omega = -\beta_{\mathrm{e}} + \Omega_{\mathrm{e}}. \end{aligned}$$

The last two equations confirm that

$$\begin{aligned} &\beta_{\mathrm{i}} = \Omega_{\mathrm{i}} + \omega, \\ &\beta_{\mathrm{e}} = \Omega_{\mathrm{e}} - \omega, \end{aligned}$$

and substituting these two relationships into the first equation gives the characteristic equation that an eigenvalue must satisfy:

$$(\Omega_{\mathrm{e}} - \omega)(\Omega_{\mathrm{i}} + \omega)\omega = \omega_{\mathrm{i}}^2(\Omega_{\mathrm{e}} - \omega) - \omega_{\mathrm{e}}^2(\Omega_{\mathrm{i}} + \omega).$$

Expanding in ω and collecting like terms gives

$$0 = \omega^3 + (\Omega_{\mathrm{i}} - \Omega_{\mathrm{e}})\omega^2 - (\Omega_{\mathrm{i}}\Omega_{\mathrm{e}} + \omega_p^2)\omega, \qquad (4)$$

where we have used that $\omega_{\mathrm{i}}^2\Omega_{\mathrm{e}} - \omega_{\mathrm{e}}^2\Omega_{\mathrm{i}} = 0$ and that the plasma frequency is $\omega_p := \sqrt{\omega_{\mathrm{i}}^2 + \omega_{\mathrm{e}}^2}$. Thus, $\omega = 0$ or

$$\omega = \tfrac{1}{2}(\Omega_{\mathrm{e}} - \Omega_{\mathrm{i}}) \pm \tfrac{1}{2}\sqrt{(\Omega_{\mathrm{e}} - \Omega_{\mathrm{i}})^2 + 4(\Omega_{\mathrm{i}}\Omega_{\mathrm{e}} + \omega_p^2)},$$

That is,

$$\boxed{\omega = \frac{\Omega_{\mathrm{e}} - \Omega_{\mathrm{i}}}{2} \pm \sqrt{\left(\frac{\Omega_{\mathrm{e}} + \Omega_{\mathrm{i}}}{2}\right)^2 + \omega_p^2}}. \qquad (5)$$

For hydrogen plasmas, $\Omega_{\mathrm{e}} \approx 1860\Omega_{\mathrm{i}}$, so

$$\omega \approx \tfrac{1}{2}\Omega_{\mathrm{e}} \pm \sqrt{(\tfrac{1}{2}\Omega_{\mathrm{e}})^2 + \omega_p^2} \quad \text{for } \Omega_{\mathrm{e}} \approx 1860\Omega_{\mathrm{i}} \text{ (hydrogen)} \qquad (6)$$

and

$$\omega = \pm\sqrt{\Omega_{\mathrm{i}}^2 + \omega_p^2} \qquad \text{for} \quad \Omega_{\mathrm{e}} = \Omega_{\mathrm{i}} \quad \text{(pair plasma)} \qquad (7)$$

Squaring and simplifying (5) yields

$$\omega^2 = \omega_p^2 + \frac{\Omega_{\mathrm{e}}^2 + \Omega_{\mathrm{i}}^2}{2} \pm (\Omega_{\mathrm{e}} - \Omega_{\mathrm{i}})\sqrt{\left(\frac{\Omega_{\mathrm{e}} + \Omega_{\mathrm{i}}}{2}\right)^2 + \omega_p^2},$$

which agrees with the cutoff limits on page 116 of Goedbloed and Poedts.[1]

[1] Hans Goedbloed and Stefaan Poedts. *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas.* Cambridge University Press, 2004.

Note that the eigenvector $\underline{v} = \begin{bmatrix} i\beta_e\beta_i \\ \omega_i\beta_e \\ \omega_e\beta_i \end{bmatrix}$ is never zero; indeed, ω_i and ω_e are strictly positive, and $\beta_i = \Omega_i + \omega$ and $\beta_e = \Omega_e - \omega$ cannot both be zero since otherwise $\Omega_i = -\omega$ and $\Omega_e = \omega$, contradicting that Ω_i and Ω_e are both strictly positive.

By the theory of Hermitian matrices, a full set of orthogonal eigenvectors must exist. Since each eigenvector has a one-dimensional eigenspace, there must be three distinct eigenvalues ω.

Decompose $\underline{v}$ into real and imaginary parts:

$$\underline{v} = \underline{a} + i\underline{b} = \begin{bmatrix} 0 \\ \omega_i\beta_e \\ \omega_e\beta_i \end{bmatrix} + i \begin{bmatrix} \beta_e\beta_i \\ 0 \\ 0 \end{bmatrix}.$$

Observe that the real and imaginary parts are orthogonal. Note that

$$-i\underline{v} = \underline{b} - i\underline{a} = \begin{bmatrix} \beta_e\beta_i \\ 0 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ \omega_i\beta_e \\ \omega_e\beta_i \end{bmatrix}$$

is also an eigenvector with eigenvalue $i\omega$ and that these two eigenvectors are orthogonal: $\underline{v}^*(-i\underline{v}) = 2\underline{a}\cdot\underline{b} = 0$. The eigenvector-eigenvalue pair $(\underline{v}, i\omega)$ corresponds to the solution

$$\begin{aligned} \underline{v}\exp(i\omega t) &= (\underline{a} + i\underline{b})(\cos\omega t + i\sin\omega t) \\ &= (\underline{a}\cos\omega t - \underline{b}\sin\omega t) + i(\underline{b}\cos\omega t + \underline{a}\sin\omega t) \\ &= \begin{bmatrix} -\beta_e\beta_i\sin\omega t \\ \omega_i\beta_e\cos\omega t \\ \omega_e\beta_i\cos\omega t \end{bmatrix} + i \begin{bmatrix} \beta_e\beta_i\cos\omega t \\ \omega_i\beta_e\sin\omega t \\ \omega_e\beta_i\sin\omega t \end{bmatrix}, \end{aligned}$$

and the eigenvector-eigenvalue pair $(-i\underline{v}, i\omega)$ corresponds to the solution

$$\begin{aligned} -i\underline{v}\exp(i\omega t) &= (\underline{b} - i\underline{a})(\cos\omega t + i\sin\omega t) \\ &= (\underline{b}\cos\omega t + \underline{a}\sin\omega t) + i(\underline{b}\cos\omega t - \underline{a}\sin\omega t) \\ &= \begin{bmatrix} \beta_e\beta_i\cos\omega t \\ \omega_i\beta_e\sin\omega t \\ \omega_e\beta_i\sin\omega t \end{bmatrix} + i \begin{bmatrix} \beta_e\beta_i\sin\omega t \\ -\omega_i\beta_e\cos\omega t \\ -\omega_e\beta_i\cos\omega t \end{bmatrix}. \end{aligned}$$

Observe that in each of these solutions the ion and electron currents are in phase and the electric field is 90 degrees out of phase relative to them.

These two solutions are independent when interpreted (in $SO(2,\mathbb{R})$) as real solutions:

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{u}}_{i1} \\ \widetilde{\mathbf{u}}_{i2} \\ \widetilde{\mathbf{u}}_{e1} \\ \widetilde{\mathbf{u}}_{e2} \end{bmatrix} = \begin{bmatrix} -\beta_e\beta_i\sin\omega t \\ \beta_e\beta_i\cos\omega t \\ \omega_i\beta_e\cos\omega t \\ \omega_i\beta_e\sin\omega t \\ \omega_e\beta_i\cos\omega t \\ \omega_e\beta_i\sin\omega t \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{u}}_{i1} \\ \widetilde{\mathbf{u}}_{i2} \\ \widetilde{\mathbf{u}}_{e1} \\ \widetilde{\mathbf{u}}_{e2} \end{bmatrix} = \begin{bmatrix} \beta_e\beta_i\cos\omega t \\ \beta_e\beta_i\sin\omega t \\ \omega_i\beta_e\sin\omega t \\ -\omega_i\beta_e\cos\omega t \\ \omega_e\beta_i\sin\omega t \\ -\omega_e\beta_i\cos\omega t \end{bmatrix}.$$

Evaluated at time 0 these solutions are

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{u}}_{i1} \\ \widetilde{\mathbf{u}}_{i2} \\ \widetilde{\mathbf{u}}_{e1} \\ \widetilde{\mathbf{u}}_{e2} \end{bmatrix} = \begin{bmatrix} 0 \\ \beta_e\beta_i \\ \omega_i\beta_e \\ 0 \\ \omega_e\beta_i \\ 0 \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{u}}_{i1} \\ \widetilde{\mathbf{u}}_{i2} \\ \widetilde{\mathbf{u}}_{e1} \\ \widetilde{\mathbf{u}}_{e2} \end{bmatrix} = \begin{bmatrix} \beta_e\beta_i \\ 0 \\ 0 \\ -\omega_i\beta_e \\ 0 \\ -\omega_e\beta_i \end{bmatrix}.$$

Note that orthogonality of complex solutions is equivalent to orthogonality of real solutions. So we have found three distinct imaginary eigenvalues and 6 orthogonal eigenvectors for the original 6×6 antisymmetric matrix.

4 Parallel system

The parallel system

$$\partial_t \begin{bmatrix} \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{u}}_{i3} \\ \widetilde{\mathbf{u}}_{e3} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_i & \omega_e \\ \omega_i & 0 & 0 \\ -\omega_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{u}}_{i3} \\ \widetilde{\mathbf{u}}_{e3} \end{bmatrix}$$

is the special, singular case of the perpendicular system (2) when the magnetic field is zero.

In this case the system (3) becomes

$$0 = (\underline{\underline{A}} - i\omega)\cdot\underline{v} = \begin{bmatrix} -i\omega & -\omega_i & \omega_e \\ \omega_i & -i\omega & 0 \\ -\omega_e & 0 & -i\omega \end{bmatrix}\cdot\underline{v}.$$

So eigenvalue/eigenvector pairs are

$$\omega = 0, \quad \underline{v} = \begin{bmatrix} 0 \\ \omega_e \\ \omega_i \end{bmatrix} \quad\text{and}\quad \omega = \pm\omega_p, \quad \underline{v} = \begin{bmatrix} i\omega \\ \omega_i \\ -\omega_e \end{bmatrix},$$

where $\omega_p := \sqrt{\omega_i^2 + \omega_e^2}$ is the plasma frequency.

To get real solutions we look at the real and imaginary parts of one of the complex-conjugate pair of solutions. Choose $\omega = \omega_p$. Write

$$\underline{a} + i\underline{b} = \begin{bmatrix} 0 \\ \omega_i \\ -\omega_e \end{bmatrix} + i \begin{bmatrix} \omega_p \\ 0 \\ 0 \end{bmatrix}.$$

Analogous to (3), the real and imaginary parts are real solutions:

$$\begin{aligned} \underline{v}\exp(i\omega_p t) &= (\underline{a} + i\underline{b})(\cos\omega_p t + i\sin\omega_p t) \\ &= (\underline{a}\cos\omega_p t - \underline{b}\sin\omega_p t) + i(\underline{b}\cos\omega_p t + \underline{a}\sin\omega_p t) \\ &= \begin{bmatrix} -\omega_p\sin\omega_p t \\ \omega_i\cos\omega_p t \\ -\omega_e\cos\omega_p t \end{bmatrix} + i \begin{bmatrix} \omega_p\cos\omega_p t \\ \omega_i\sin\omega_p t \\ -\omega_e\sin\omega_p t \end{bmatrix}. \end{aligned}$$

So three orthogonal eigensolutions are

$$\begin{bmatrix} 0 \\ \omega_e \\ \omega_i \end{bmatrix}, \qquad \begin{bmatrix} -\omega_p\sin\omega_p t \\ \omega_i\cos\omega_p t \\ -\omega_e\cos\omega_p t \end{bmatrix}, \qquad \begin{bmatrix} \omega_p\cos\omega_p t \\ \omega_i\sin\omega_p t \\ -\omega_e\sin\omega_p t \end{bmatrix}.$$

Evaluated at time 0 these solutions are

$$\begin{bmatrix} 0 \\ \omega_e \\ \omega_i \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ \omega_i \\ -\omega_e \end{bmatrix}, \qquad \begin{bmatrix} \omega_p \\ 0 \\ 0 \end{bmatrix}.$$