Fast magnetic reconnection with a ten-moment two-fluid plasma model

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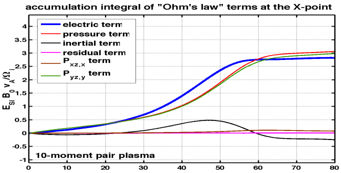
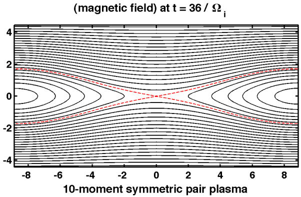


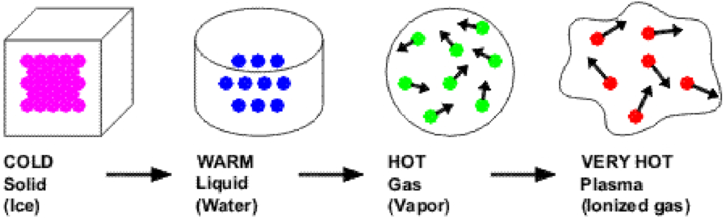


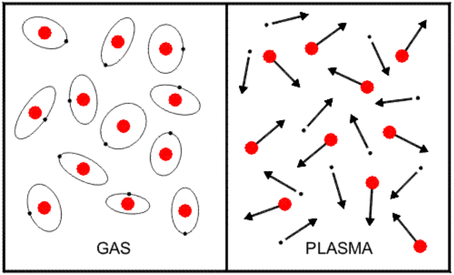
Outline

1. Plasma modeling
2. Magnetic reconnection
3. Heat flux

Part I: Plasma modeling

Fluids and plasmas





Model hierarchy

(Boltzmann-Maxwell)

↓

(10-moment Maxwell) ⟶ (10-moment two-fluid MHD)

↓ ↓

(5-moment Maxwell) ⟶ (5-moment two-fluid MHD) ⟶ MHD

Two-species models: ions, electrons (s=i,e)

Boltzmann-Maxwell model

- **Boltzmann equations:**

$$\partial_t f_i + \mathbf{v}\cdot\nabla_x f_i + \mathbf{a}_\mathrm{i}\cdot\nabla_\mathbf{v} f_i = C_i + C_{ie},$$
$$\partial_t f_e + \mathbf{v}\cdot\nabla_x f_e + \mathbf{a}_\mathrm{e}\cdot\nabla_\mathbf{v} f_e = C_e + C_{ei},$$

- **Lorentz force law**

$$\mathbf{a}_\mathrm{i} = \tfrac{q_i}{m_i}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right),$$
$$\mathbf{a}_\mathrm{e} = \tfrac{q_e}{m_e}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

- **Maxwell's equations:**

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} + \nabla\times\begin{bmatrix}\mathbf{E}\\ -c^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}0\\ -\mathbf{J}/\epsilon_0\end{bmatrix},$$
$$\nabla\cdot\mathbf{B} = 0,\quad \nabla\cdot\mathbf{E} = \sigma/\epsilon_0,$$
$$\sigma = \sum_\mathrm{s}\frac{q_\mathrm{s}}{m_\mathrm{s}}\int f_\mathrm{s}\,d\mathbf{v},\quad \mathbf{J} = \sum_\mathrm{s}\frac{q_\mathrm{s}}{m_\mathrm{s}}\int \mathbf{v} f_\mathrm{s}\,d\mathbf{v}$$

10-moment two-fluid Maxwell model:

- moments:

$$\begin{bmatrix}\rho_\mathrm{s}\\ \rho_\mathrm{s}\mathbf{u}_\mathrm{s}\\ \mathbb{E}_\mathrm{s}\end{bmatrix} = \int\begin{bmatrix}1\\ \mathbf{v}\\ \mathbf{vv}\end{bmatrix} f_\mathrm{s}\,d\mathbf{v}$$

- closure:

$$\mathbb{R}_\mathrm{s} = \int \mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s}\,C_\mathrm{s}\,d\mathbf{v},$$
$$\begin{bmatrix}\mathbf{R}_\mathrm{s}\\ \mathbb{Q}_\mathrm{s}\end{bmatrix} = \int\begin{bmatrix}\mathbf{v}\\ \mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s}\end{bmatrix} C_{sp}\,d\mathbf{v},$$
$$\mathfrak{q}_\mathrm{s} = \int \mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s}\,f_\mathrm{s}\,d\mathbf{c}_\mathrm{s}$$
$$(\mathbf{c}_\mathrm{s} := \mathbf{v} - \mathbf{u}_\mathrm{s})$$

Boltzmann-Maxwell model

- **Boltzmann equations:**

$$\partial_t f_i + \mathbf{v}\cdot\nabla_x f_i + \mathbf{a}_\mathrm{i}\cdot\nabla_\mathbf{v} f_i = C_i + C_{ie},$$
$$\partial_t f_e + \mathbf{v}\cdot\nabla_x f_e + \mathbf{a}_\mathrm{e}\cdot\nabla_\mathbf{v} f_e = C_e + C_{ei},$$

- **Lorentz force law**

$$\mathbf{a}_\mathrm{i} = \frac{q_i}{m_i}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right),$$
$$\mathbf{a}_\mathrm{e} = \frac{q_e}{m_e}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

- **Maxwell's equations:**

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{B}\\ \mathbf{E}\end{bmatrix} + \nabla\times\begin{bmatrix}\mathbf{E}\\ -c^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}0\\ -\mathbf{J}/\epsilon_0\end{bmatrix},$$
$$\nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{E}=\sigma/\epsilon_0,$$
$$\sigma = \sum_\mathrm{s}\frac{q_\mathrm{s}}{m_\mathrm{s}}\int f_\mathrm{s}\,d\mathbf{v},\quad \mathbf{J} = \sum_\mathrm{s}\frac{q_\mathrm{s}}{m_\mathrm{s}}\int \mathbf{v} f_\mathrm{s}\,d\mathbf{v}$$

5-moment two-fluid Maxwell model:

- moments:

$$\begin{bmatrix}\rho_\mathrm{s}\\ \rho_\mathrm{s}\mathbf{u}_\mathrm{s}\\ \mathcal{E}_\mathrm{s}\end{bmatrix} = \int\begin{bmatrix}1\\ \mathbf{v}\\ \frac{1}{2}\|\mathbf{v}\|^2\end{bmatrix} f_\mathrm{s}\,d\mathbf{v}$$

- closure:

$$\mathbb{P}_\mathrm{s}^\circ = \int\left(\mathbf{c}_\mathrm{s}\mathbf{c}_\mathrm{s} - \|\mathbf{c}_\mathrm{s}\|^2\mathbb{1}/3\right) f_\mathrm{s}\,d\mathbf{v},$$
$$\begin{bmatrix}\mathbf{R}_\mathrm{s}\\ Q_\mathrm{s}\end{bmatrix} = \int\begin{bmatrix}\mathbf{v}\\ \frac{1}{2}\|\mathbf{c}_\mathrm{s}\|^2\end{bmatrix} C_{sp}\,d\mathbf{v},$$
$$\mathbf{q}_\mathrm{s} = \int \frac{1}{2}\mathbf{c}_\mathrm{s}\|\mathbf{c}_\mathrm{s}\|^2 f_\mathrm{s}\,d\mathbf{v}$$
$$(\mathbf{c}_\mathrm{s} := \mathbf{v}-\mathbf{u}_\mathrm{s})$$

Equations of the 10-moment 2-fluid Maxwell model

Gas dynamics equations

$$\bar{\delta}_t \begin{bmatrix} \rho_{\rm i} \\ \rho_{\rm i}\mathbf{u}_{\rm i} \\ \rho_{\rm i}\mathbb{e}_{\rm i} \end{bmatrix} + \begin{bmatrix} 0 \\ \nabla\cdot\mathbb{P}_{\rm i} \\ \mathrm{Sym2}(\nabla\cdot(\mathbb{P}_{\rm i}\mathbf{u}_{\rm i})) + \nabla\cdot\mathbb{q}_{\rm i} \end{bmatrix} = \sigma_{\rm i}\begin{bmatrix} 0 \\ \mathbf{E}+\mathbf{u}_{\rm i}\times\mathbf{B} \\ \mathrm{Sym2}(\mathbf{u}_{\rm i}\mathbf{E}+\mathbb{e}_{\rm i}\times\mathbf{B}) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{R}_{\rm i} \\ \mathbb{R}_{\rm i}+\mathbb{Q}_{\rm i} \end{bmatrix}$$

$$\bar{\delta}_t \begin{bmatrix} \rho_{\rm e} \\ \rho_{\rm e}\mathbf{u}_{\rm e} \\ \rho_{\rm e}\mathbb{e}_{\rm e} \end{bmatrix} + \begin{bmatrix} 0 \\ \nabla\cdot\mathbb{P}_{\rm e} \\ \mathrm{Sym2}(\nabla\cdot(\mathbb{P}_{\rm e}\mathbf{u}_{\rm e})) + \nabla\cdot\mathbb{q}_{\rm e} \end{bmatrix} = \sigma_{\rm e}\begin{bmatrix} 0 \\ \mathbf{E}+\mathbf{u}_{\rm e}\times\mathbf{B} \\ \mathrm{Sym2}(\mathbf{u}_{\rm e}\mathbf{E}+\mathbb{e}_{\rm e}\times\mathbf{B}) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{R}_{\rm e} \\ \mathbb{R}_{\rm e}+\mathbb{Q}_{\rm e} \end{bmatrix}$$

where $\bar{\delta}_t\alpha := \partial_t\alpha + \nabla\cdot(\mathbf{u}_{\rm s}\alpha)$,

Maxwell's equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix} + \nabla\times\begin{bmatrix} \mathbf{E} \\ -c^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -c^2\mathbf{J} \end{bmatrix},$$

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\cdot\mathbf{E} = \sigma/\epsilon_0,$$

$$\sigma_{\rm s} = \frac{q_{\rm s}}{m_{\rm s}}\rho_{\rm s}, \quad \sigma = \sum_s \sigma_{\rm s}, \quad \mathbf{J} = \sum_s \sigma_{\rm s}\mathbf{u}_s$$

Closures:

$$\mathbb{R}_{\rm s} = -\tau_{\rm s}^{-1}\mathbb{P}_{\rm s}^{\circ}$$

$$\mathbb{q}_{\rm s} = -\frac{2}{5}\mathbf{K}_{\rm s} \vdots \mathrm{Sym3}\left(\frac{\mathbb{T}_{\rm s}}{T_{\rm s}}\cdot\nabla\mathbb{T}_{\rm s}\right)$$

$$-\mathbf{R}_{\rm i} = \mathbf{R}_{\rm e} = ne\eta\cdot\mathbf{J}\ ?$$

$$\mathbb{Q}_{\rm s} = ?$$

Equations of the 5-moment 2-fluid Maxwell model

Gas dynamics equations

$$\bar{\delta}_t \begin{bmatrix} \rho_\mathrm{i} \\ \rho_\mathrm{i}\mathbf{u}_\mathrm{i} \\ \rho_\mathrm{i}\mathrm{e}_\mathrm{i} \end{bmatrix} + \begin{bmatrix} 0 \\ \nabla p_\mathrm{i} + \nabla\cdot\mathbb{P}_\mathrm{i}^\circ \\ \nabla\cdot(\mathbf{u}_\mathrm{i} p_\mathrm{i}) + \nabla\cdot(\mathbf{u}_\mathrm{i}\cdot\mathbb{P}_\mathrm{i}^\circ) + \nabla\cdot\mathbf{q}_\mathrm{i} \end{bmatrix} = \sigma_\mathrm{i} \begin{bmatrix} 0 \\ \mathbf{E} + \mathbf{u}_\mathrm{i}\times\mathbf{B} \\ \mathbf{u}_\mathrm{i}\cdot\mathbf{E} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{R}_\mathrm{i} \\ Q_\mathrm{i} \end{bmatrix}$$

$$\bar{\delta}_t \begin{bmatrix} \rho_\mathrm{e} \\ \rho_\mathrm{e}\mathbf{u}_\mathrm{e} \\ \rho_\mathrm{e}\mathrm{e}_\mathrm{e} \end{bmatrix} + \begin{bmatrix} 0 \\ \nabla p_\mathrm{e} + \nabla\cdot\mathbb{P}_\mathrm{e}^\circ \\ \nabla\cdot(\mathbf{u}_\mathrm{e} p_\mathrm{e}) + \nabla\cdot(\mathbf{u}_\mathrm{e}\cdot\mathbb{P}_\mathrm{e}^\circ) + \nabla\cdot\mathbf{q}_\mathrm{e} \end{bmatrix} = \sigma_\mathrm{e} \begin{bmatrix} 0 \\ \mathbf{E} + \mathbf{u}_\mathrm{e}\times\mathbf{B} \\ \mathbf{u}_\mathrm{i}\cdot\mathbf{E} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{R}_\mathrm{e} \\ Q_\mathrm{e} \end{bmatrix}$$

where $\bar{\delta}_t\alpha := \partial_t\alpha + \nabla\cdot(\mathbf{u}_\mathrm{s}\alpha)$,

Maxwell's equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{B} \\ \mathbf{E} \end{bmatrix} + \nabla\times\begin{bmatrix} \mathbf{E} \\ -c^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -c^2\mathbf{J} \end{bmatrix},$$

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\cdot\mathbf{E} = \sigma/\epsilon_0,$$

$$\sigma_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}}\rho_\mathrm{s}, \quad \sigma = \sum_s \sigma_\mathrm{s}, \quad \mathbf{J} = \sum_s \sigma_\mathrm{s}\mathbf{u}_s$$

Closures:

$$\mathbb{P}_\mathrm{s}^\circ = -2\mu : (\nabla\mathbf{u})^\circ$$

$$\mathbb{q}_\mathrm{s} = -\mathbf{k}\cdot\nabla T$$

$$-\mathbf{R}_\mathrm{i} = \mathbf{R}_\mathrm{e} = ne\eta\cdot\mathbf{J}?$$

$$Q_\mathrm{s} = ?$$

MHD: Maxwell's equations

Magnetohydrodyamics (MHD) assumes that the light speed is infinite. Then Maxwell's equations simplify to

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \cancel{c^{-2}\partial_t \mathbf{E}}, \qquad \mu_0 \sigma = 0 + \cancel{c^{-2}\nabla \cdot \mathbf{E}}$$

This system is Galilean-invariant, but its relationship to gas-dynamics is fundamentally different:

variable	**MHD**	**2-fluid-Maxwell**
$\mathbf{E}$	supplied by *Ohm's law* (from gas dynamics)	evolved (from $\mathbf{B}$ and $\mathbf{J}$)
$\mathbf{J}$	$\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ (comes from $\mathbf{B}$)	$\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$ (from gas dynamics)
σ	$\sigma = 0$ (quasineutrality) (gas-dynamic constraint)	$\sigma = e(n_i - n_e)$ (electric field constraint)

MHD: charge neutrality

The assumption of charge neutrality reduces the number of gas-dynamic equations that must be solved:

- ***net* density** evolution

  The density of each species is the same:

  $$n_\mathrm{i} = n_\mathrm{e} = n$$

- ***net* velocity** evolution

  The species fluid velocities can be inferred from the net current, net velocity, and density:

  $$\mathbf{u}_\mathrm{i} = \mathbf{u} + \frac{m_\mathrm{e}}{m_\mathrm{i} + m_\mathrm{e}} \frac{\mathbf{J}}{ne}, \qquad \mathbf{u}_\mathrm{e} = \mathbf{u} - \frac{m_\mathrm{i}}{m_\mathrm{i} + m_\mathrm{e}} \frac{\mathbf{J}}{ne}.$$

MHD: Ohm's law

Ohm's law is current evolution solved for the electric field:

$$
\begin{aligned}
\mathbf{E} =& \mathbf{B} \times \mathbf{u} && \text{(ideal term)} \\
&+ \boldsymbol{\eta} \cdot \mathbf{J} && \text{(resistive term)} \\
&+ \tfrac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B} && \text{(Hall term)} \\
&+ \tfrac{1}{e\rho} \nabla \cdot (m_\mathrm{e} \mathbb{P}_\mathrm{i} - m_\mathrm{i} \mathbb{P}_\mathrm{e}) && \text{(pressure term)} \\
&+ \tfrac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \tfrac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right] && \text{(inertial term).}
\end{aligned}
$$

Ohm's law gives an implicit closure to the induction equation, $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$, and entails an implicit numerical method.

Pressure evolution

$$n_\mathrm{i} d_t \mathbb{T}_\mathrm{i} + \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \cdot \nabla \mathbf{u}_\mathrm{i}) + \nabla \cdot \mathbb{q}_\mathrm{i} = \frac{q_\mathrm{i}}{m_\mathrm{i}} \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \times \mathbf{B}) + \mathbb{R}_\mathrm{i} + \mathbb{Q}_\mathrm{i},$$

$$n_\mathrm{e} d_t \mathbb{T}_\mathrm{e} + \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \cdot \nabla \mathbf{u}_\mathrm{e}) + \nabla \cdot \mathbb{q}_\mathrm{e} = \frac{q_\mathrm{e}}{m_\mathrm{e}} \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \times \mathbf{B}) + \mathbb{R}_\mathrm{e} + \mathbb{Q}_\mathrm{e}.$$

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B},$$

Ohm's law

$$\mathbf{E} = \eta \cdot \mathbf{J} + \mathbf{B} \times \mathbf{u} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B}$$
$$+ \frac{1}{e\rho} \nabla \cdot (m_\mathrm{e} \mathbb{P}_\mathrm{i} - m_\mathrm{i} \mathbb{P}_\mathrm{e})$$
$$+ \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right]$$

Definitions:

$$d_t := \partial_t + \mathbf{u}_\mathrm{s} \cdot \nabla,$$

$$\mathbb{P}^\mathrm{d} := \rho_i \mathbf{w}_\mathrm{i} \mathbf{w}_\mathrm{i} + \rho_e \mathbf{w}_\mathrm{e} \mathbf{w}_e$$

$$\mathbf{w}_\mathrm{i} = \frac{m_\mathrm{e} \mathbf{J}}{e\rho}, \quad \mathbf{w}_\mathrm{e} = -\frac{m_\mathrm{i} \mathbf{J}}{e\rho}$$

Closures:

$$\mathbb{R}_\mathrm{s} = -\tau_\mathrm{s}^{-1} \mathbb{P}_\mathrm{s}^\circ$$

$$\mathbb{q}_\mathrm{s} = -\frac{2}{5} \mathbf{K}_\mathrm{s} : \mathrm{Sym3}\left(\frac{\mathbb{T}_\mathrm{s}}{T_\mathrm{s}} \cdot \nabla \mathbb{T}_\mathrm{s} \right)$$

$$-\mathbf{R}_\mathrm{i} = \mathbf{R}_\mathrm{e} = ne\eta \cdot \mathbf{J}$$

$$\mathbb{Q}_\mathrm{s} = ?$$

Equations of 5-moment 2-fluid MHD

Pressure evolution

$$\tfrac{3}{2} n d_t T_\mathrm{i} + p_\mathrm{i} \nabla \cdot \mathbf{u}_\mathrm{i} + \mathbb{P}_\mathrm{i}^\circ : \nabla \mathbf{u}_\mathrm{i} + \nabla \cdot \mathbf{q}_\mathrm{i} = Q_\mathrm{i},$$
$$\tfrac{3}{2} n d_t T_\mathrm{e} + p_\mathrm{e} \nabla \cdot \mathbf{u}_\mathrm{e} + \mathbb{P}_\mathrm{e}^\circ : \nabla \mathbf{u}_\mathrm{e} + \nabla \cdot \mathbf{q}_\mathrm{e} = Q_\mathrm{e};$$

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$
$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B},$$

Ohm's law

$$\mathbf{E} = \boldsymbol{\eta} \cdot \mathbf{J} + \mathbf{B} \times \mathbf{u} + \tfrac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B}$$
$$+ \tfrac{1}{e\rho} \nabla \cdot \left(m_\mathrm{e}(p_\mathrm{i} + \mathbb{P}_\mathrm{i}^\circ) - m_\mathrm{i}(p_\mathrm{e} + \mathbb{P}_\mathrm{e}^\circ) \right)$$
$$+ \tfrac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \tfrac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right]$$

Definitions:

$$d_t := \partial_t + \mathbf{u}_\mathrm{s} \cdot \nabla,$$
$$\mathbb{P}^\mathrm{d} := \rho_i \mathbf{w}_\mathrm{i} \mathbf{w}_\mathrm{i} + \rho_e \mathbf{w}_\mathrm{e} \mathbf{w}_e$$
$$\mathbf{w}_\mathrm{i} = \frac{m_\mathrm{e} \mathbf{J}}{e\rho}, \quad \mathbf{w}_\mathrm{e} = -\frac{m_\mathrm{i} \mathbf{J}}{e\rho}$$

Closures:

$$\mathbb{P}_\mathrm{s}^\circ = -2\boldsymbol{\mu} : (\nabla \mathbf{u})^\circ$$
$$\mathbb{q}_\mathrm{s} = -\mathbf{k} \cdot \nabla T$$
$$-\mathbf{R}_\mathrm{i} = \mathbf{R}_\mathrm{e} = ne\boldsymbol{\eta} \cdot \mathbf{J}?$$
$$Q_\mathrm{s} = ?$$

Resistive MHD

Ohm's law:

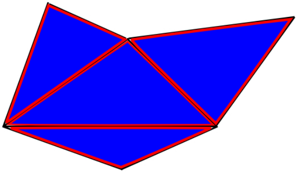
$$
\begin{aligned}
\mathbf{E} =& \eta \cdot \mathbf{J} && \text{(resistive term)} \\
&+ \mathbf{B} \times \mathbf{u} && \text{(ideal term)} \\
&+ \tfrac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B} && \text{(Hall term)} \\
&+ \tfrac{1}{e\rho} \nabla \cdot \left(m_\mathrm{e} \mathbb{P}_\mathrm{i} - m_\mathrm{i} \mathbb{P}_\mathrm{e} \right) && \text{(pressure term)} \\
&+ \tfrac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \tfrac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right] && \text{(inertial term).}
\end{aligned}
$$

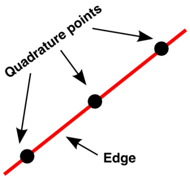
Resistive MHD model:

$$
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ \mathcal{E} \\ \mathbf{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{u} \\ \rho \mathbf{uu} + \left(p + \frac{1}{2} \|\mathbf{B}\|^2 \right) \mathbb{I} - \mathbf{BB} \\ \mathbf{u} \left(\mathcal{E} + p + \frac{1}{2} \|\mathbf{B}\|^2 \right) - \mathbf{B} \left(\mathbf{u} \cdot \mathbf{B} \right) \\ \mathbf{uB} - \mathbf{Bu} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \\ \eta \nabla \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})] \\ \eta \nabla^2 \mathbf{B} \end{bmatrix}
$$

$$
\nabla \cdot \mathbf{B} = 0
$$

Discontinuous Galerkin methods





Spatial discretization [Cockburn & Shu, 1990's]

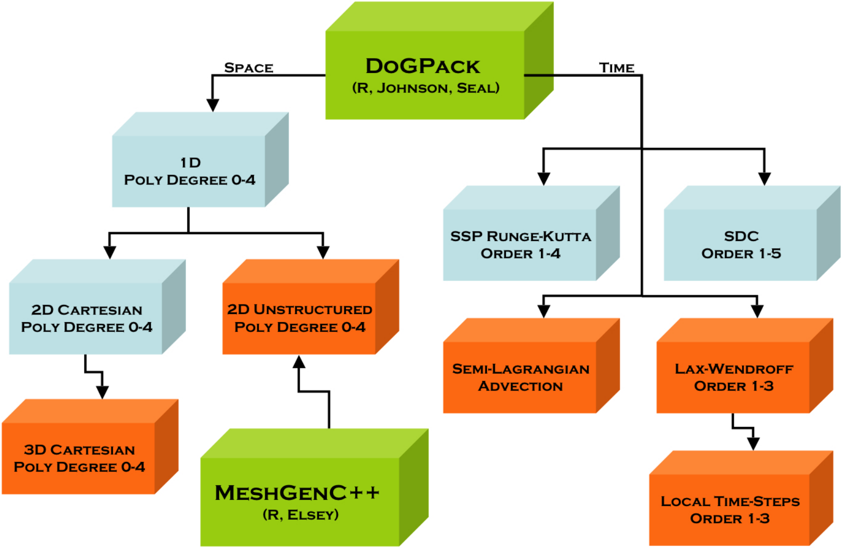
- Basis functions: $\phi^{(\ell)}(\mathbf{x}) = \{\ldots, \quad x^{k-1}, \quad x^{k-2}\,y, \quad \ldots, \quad x\,y^{k-2}, \quad y^{k-1}\}$
- Galerkin expansion: $q^h(\mathbf{x}, t) = \sum_{n=1}^{k(k+1)/2} Q^{(\ell)}(t)\,\phi^{(\ell)}(\mathbf{x})$
- $\forall \mathcal{T}$ start with $q_{,t} + \nabla \cdot \mathbf{F}(q) = 0$ and obtain semi-discrete weak-form:

$$\int_{\mathcal{T}} \phi^{(\ell)}\, q_{,t}\, d\mathbf{x} = -\int_{\mathcal{T}} \phi^{(\ell)}\, \nabla \cdot \mathbf{F}(q)\, d\mathbf{x}$$

$$\Longrightarrow \quad \frac{d}{dt} Q^{(\ell)} = \underbrace{\frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} \nabla\phi^{(\ell)} \cdot \mathbf{F}(q)\, d\mathbf{x}}_{\textit{Interior}} - \underbrace{\frac{1}{|\mathcal{T}|} \oint_{\partial\mathcal{T}} \phi^{(\ell)}\, \mathbf{F}(q) \cdot d\mathbf{s}}_{\textit{Edge}}$$

- Interior: numerical quadrature, Edge: approx Riemann soln, then quadrature

The DoGPack software package

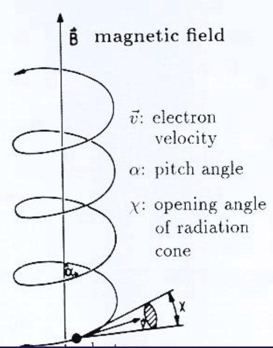


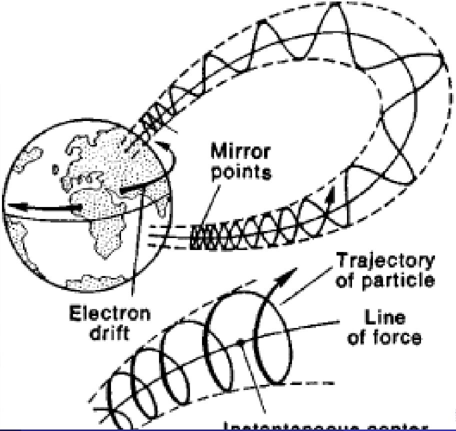
Part II: Magnetic reconnection

“Frozen-in” magnetic field lines

How does a plasma act differently from a normal gas?

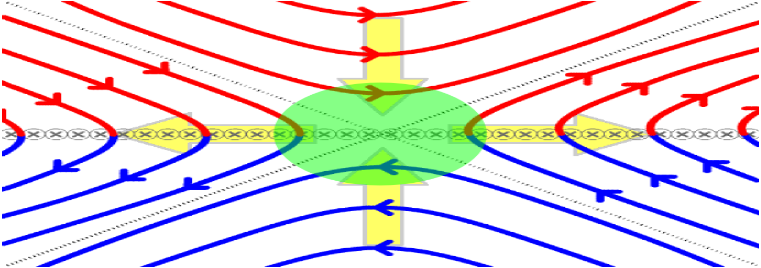
- moving charges $\longrightarrow$ electrical current $\longrightarrow$ magnetic field
- charged particles spiral around magnetic field lines.
- viewed from a distance, the particles are stuck to the field lines.
- so magnetic field lines approximately move with the plasma.





Magnetic Reconnection

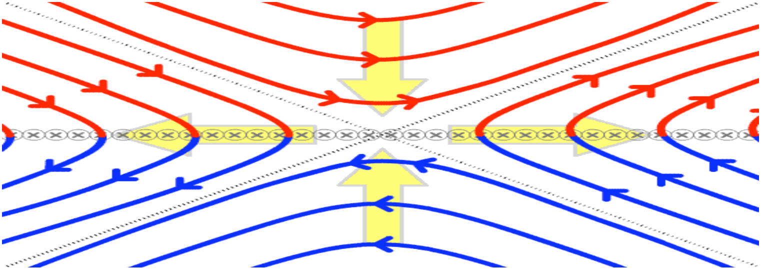
- **Start:** oppositely directed field lines are driven towards each other.
- Field lines reconnect at the **X-point**.
- **Lower energy state:** change topology of field lines
- Results in large energy release in the form of oppositely directed jets



2D separator reconnection

Magnetic Reconnection

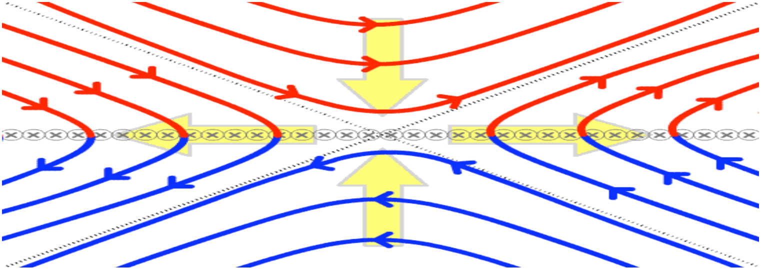
- **Start:** oppositely directed field lines are driven towards each other.
- Field lines reconnect at the **X-point**.
- **Lower energy state:** change topology of field lines
- Results in large energy release in the form of oppositely directed jets



2D separator reconnection

Magnetic Reconnection

- **Start:** oppositely directed field lines are driven towards each other.
- Field lines reconnect at the **X-point**.
- **Lower energy state:** change topology of field lines
- Results in large energy release in the form of oppositely directed jets



2D separator reconnection

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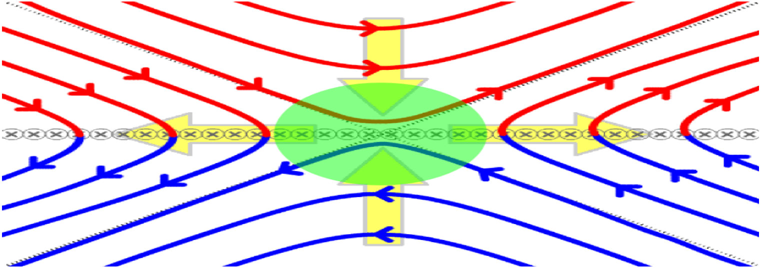

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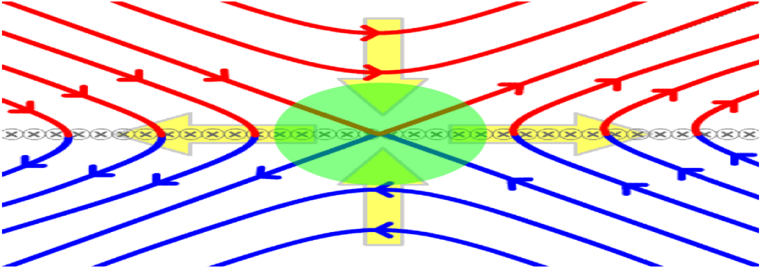

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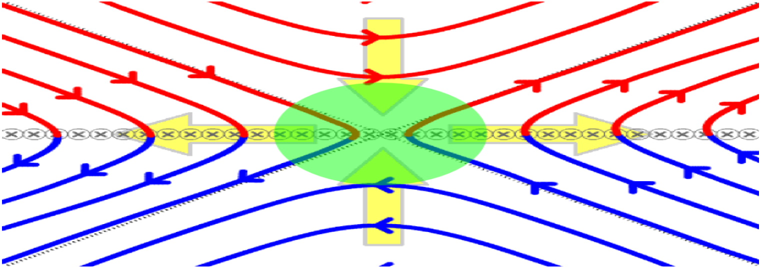

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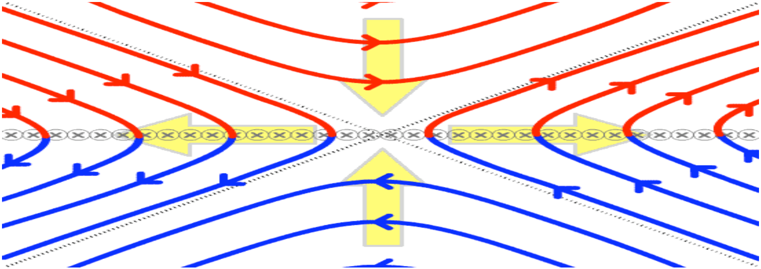

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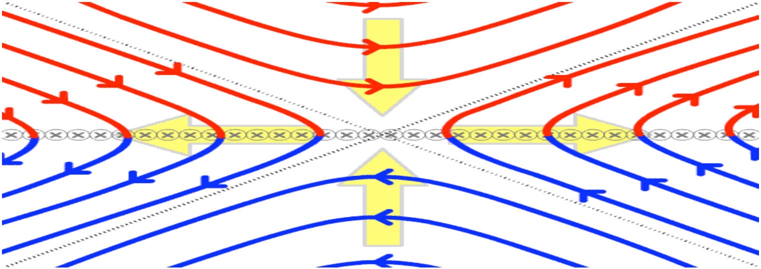

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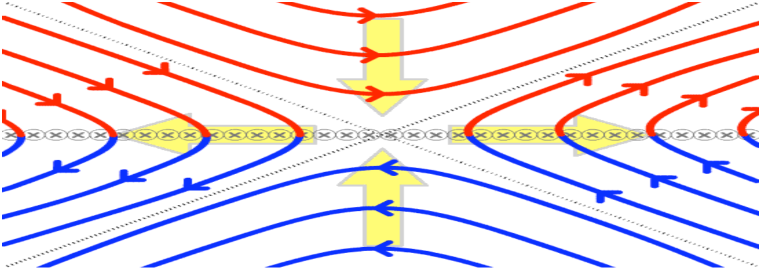


2D separator reconnection

X-point analysis

At the X-point the momentum equation ("Ohm's law") reduces to

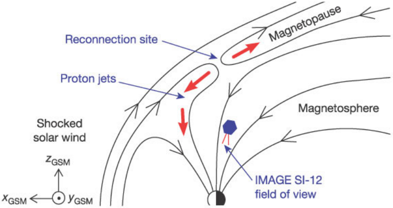
$$\text{rate of reconnection} = \mathbf{E}_3(0) = \frac{-\mathbf{R}_\mathrm{i}}{e n_\mathrm{i}} \quad \text{(resistive term)}$$
$$+ \frac{\nabla \cdot \mathbb{P}_\mathrm{i}}{e n_\mathrm{i}} \quad \text{(pressure term)}$$
$$+ \frac{m_\mathrm{i}}{e} \partial_t \mathbf{u}_\mathrm{i} \quad \text{(inertial term)}$$

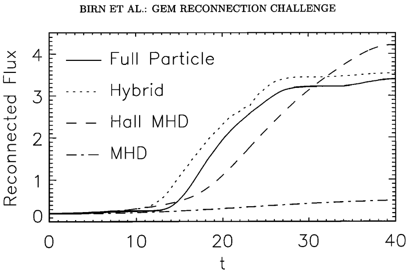
Consequences:

1. *Collisionless reconnection* is supported by the inertial or pressure term.
2. For the 5-moment model the *inertial* term must support the reconnection; i.e. each species velocity at the origin should track exactly with reconnected flux.
3. For *steady-state* reconnection without resistivity the *pressure* term must provide for the reconnection.

Collisionless magnetic reconnection: GEM problem

[Frey et al., Nature, 2003]



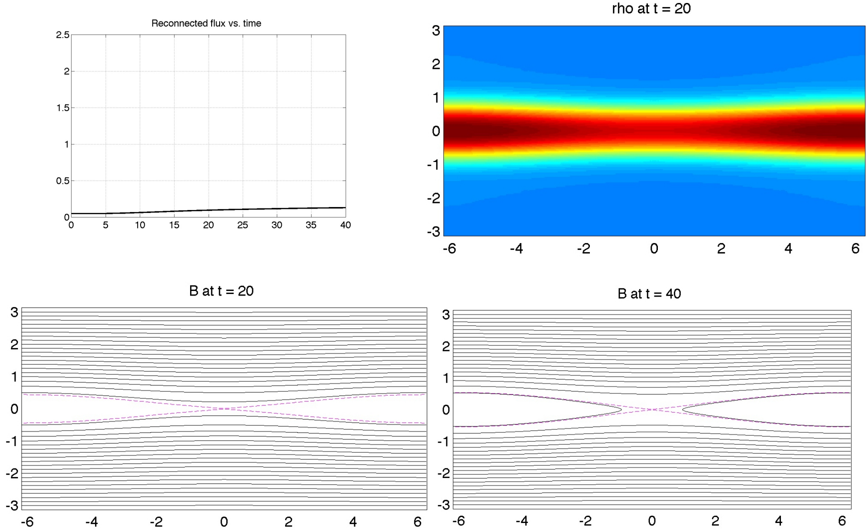


1. **[Shay et al., 2001]**: GEM challenge problem studied reconnection rate for different models; concluded that the Hall term is critical: $(m_i - m_e)\mathbf{J}\times\mathbf{B}/(\rho e)$
2. **[Bessho and Bhattacharjee, 2007]**: fast reconnection in electron-positron plasma; Hall term is absent, dominant term in Ohm's law is $\nabla\cdot\mathbb{P}$
3. **[Chacón et al., 2008]**: Fluid case: steady fast reconnection in a five-moment viscous magnetized pair plasma

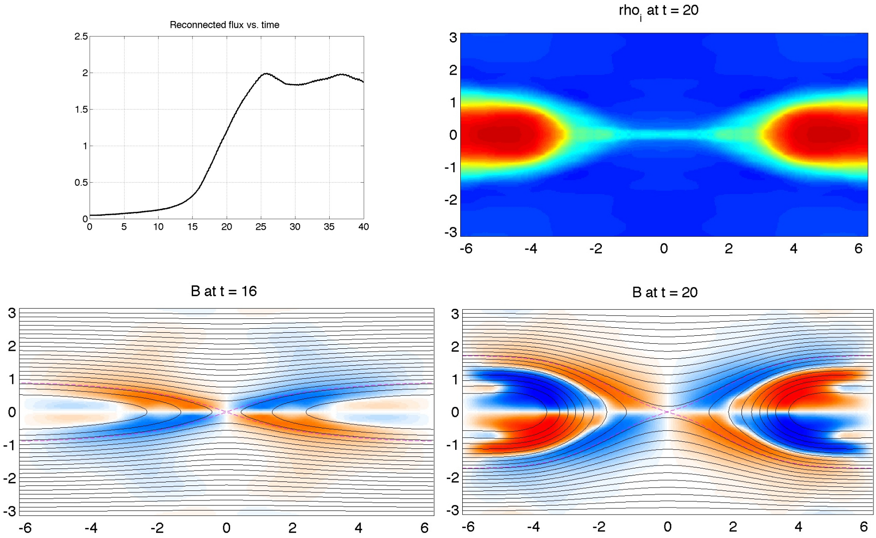
Q1: fast reconnection in an electron-positron plasma with only scalar pressure?

Q2: fast reconnection in an electron-positron plasma with 10-moment model?

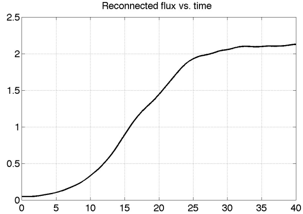
GEM: Resistive MHD ($\eta = 5 \times 10^{-3}$)

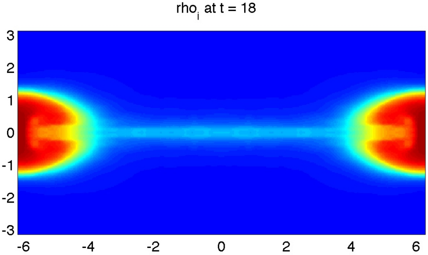


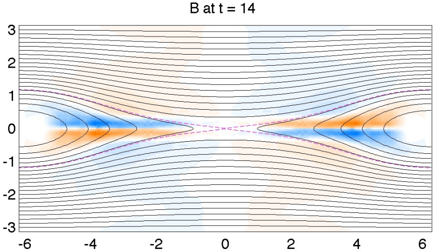
GEM: 2-fluid 5-moment ($\frac{m_i}{m_e} = 25$)

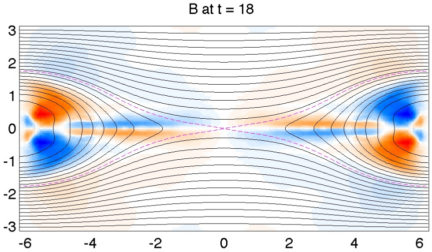


GEM: 2-fluid 5-moment ($\frac{m_i}{m_e} = 1$)

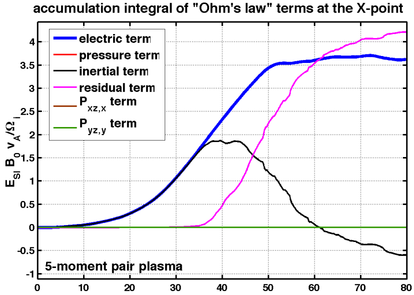


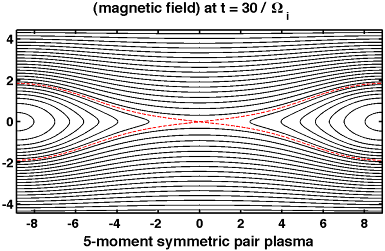


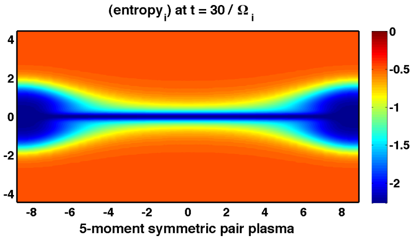


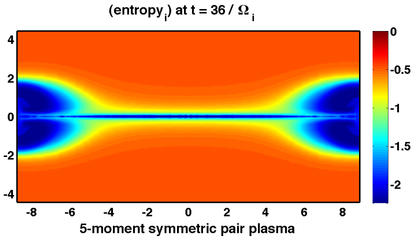


GEM: 2-fluid 5-moment ($\frac{m_i}{m_e} = 1,\ \tau = 0$)

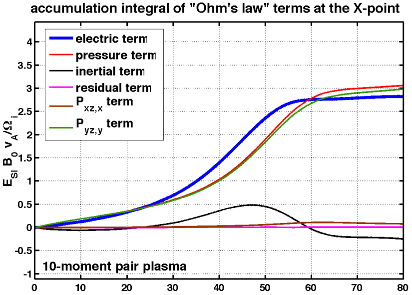


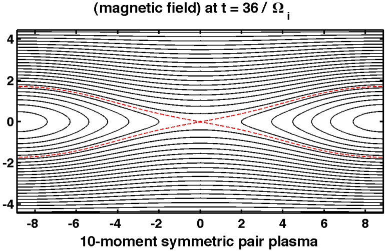


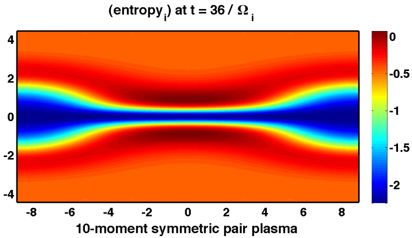


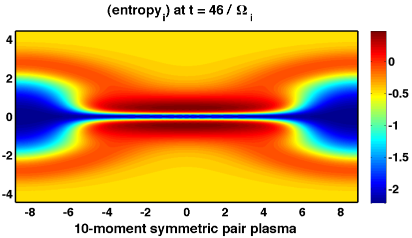


GEM: 2-fluid 10-moment ($\frac{m_i}{m_e} = 1$, $\tau = 0.2$)









How does our model compare with published kinetic simulations on the full GEM problem?

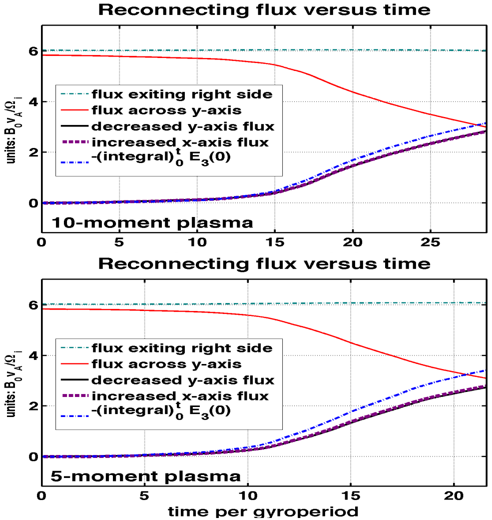
We compare the time until peak reconnection rate with published kinetic results.

model	16% flux reconnected
Vlasov [ScGr06]	$t = 17.7/\Omega_i$:
PIC [Pritchet01]	$t = 15.7/\Omega_i$:
10-moment	$t = 18/\Omega_i$:
5-moment	$t = 13.5/\Omega_i$:

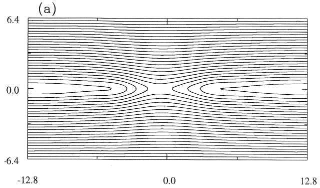
GEM ($\frac{m_i}{m_e} = 25$): kinetic models vs. 5- and 10-moment

The ten-moment model attained 16% flux reconnected at about $t = 18/\Omega_i$:

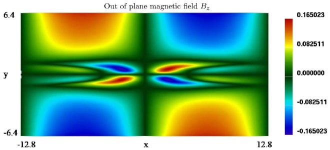
The five-moment model attained 16% flux reconnected at about $t = 13.5/\Omega_i$:



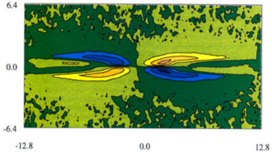
Magnetic field at 16% reconnected



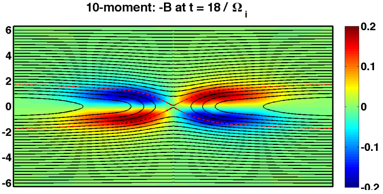
Magnetic field lines for PIC at $\Omega_i t = 15.7$ [Pritchett01]

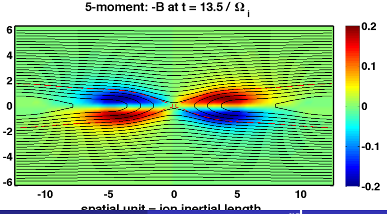


Magnetic field for Vlasov at $\Omega_i t = 17.7$ [ScGr06]

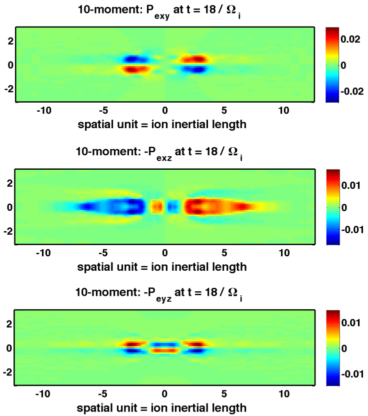


magnetic field of [Pritchett01]

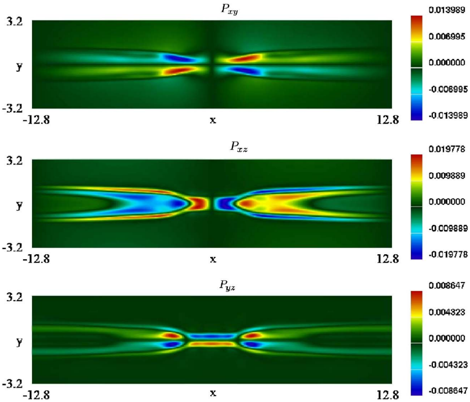




Off-diagonal components of electron pressure tensor

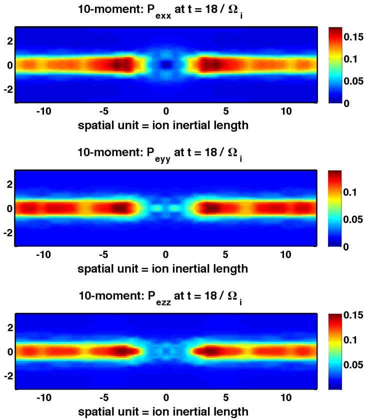


Off-diagonal components of the electron pressure tensor for 10-moment simulation at $\Omega_i t = 18$

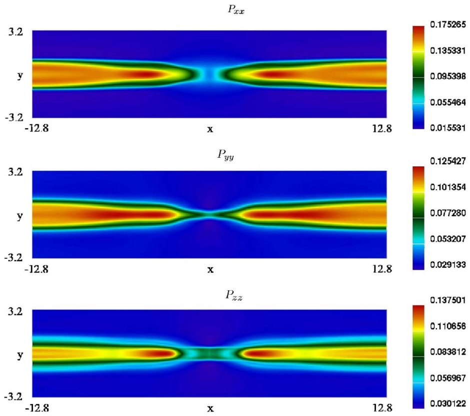


Off-diagonal components of the electron pressure tensor for Vlasov simulation at $\Omega_i t = 17.7$ [ScGr06]

Diagonal components of electron pressure tensor



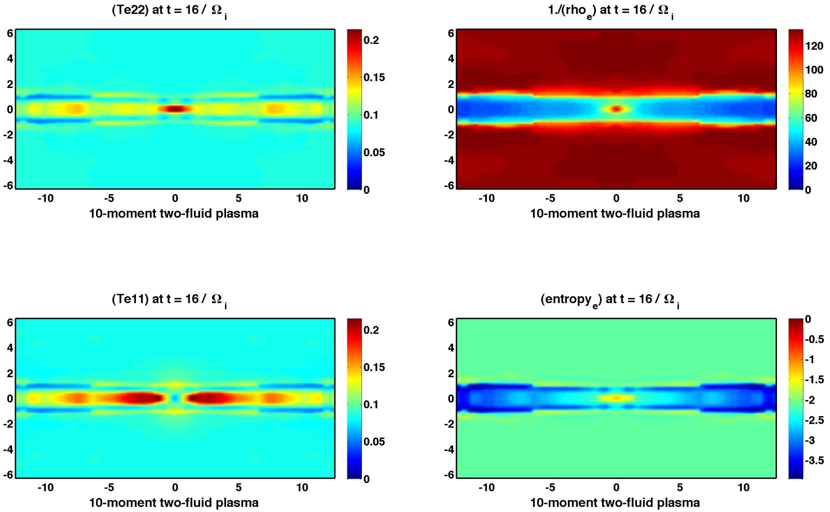
Diagonal components of the electron pressure tensor for 10-moment simulation at $\Omega_i t = 18$



Diagonal components of the electron pressure tensor for Vlasov simulation at $\Omega_i t = 17.7$ [ScGr06]

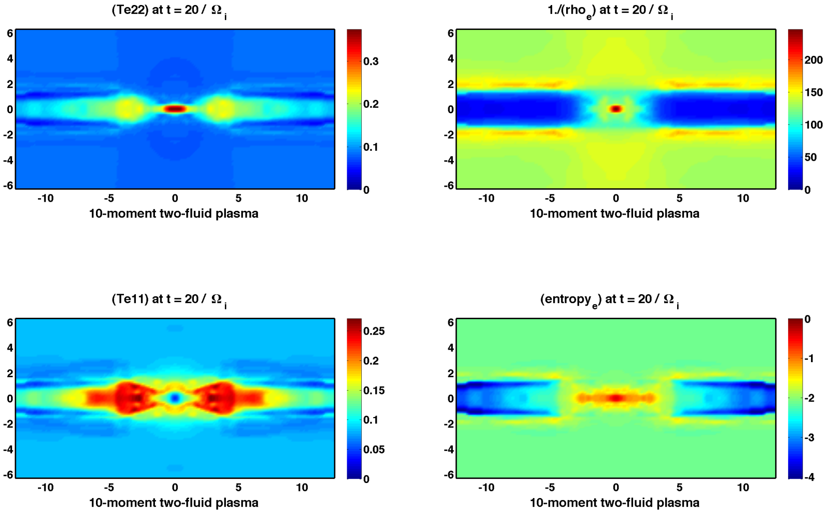
Electron gas at $t = 16$

Problem: the code crashes! Why? Look at electron gas dynamics:



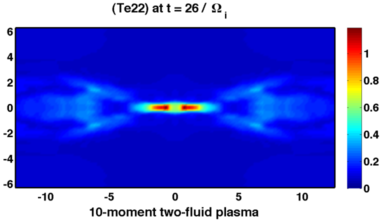
Electron gas at $t = 20$

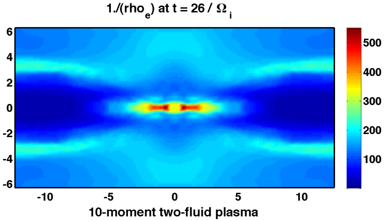
Problem: the code crashes! Why? Look at electron gas dynamics:

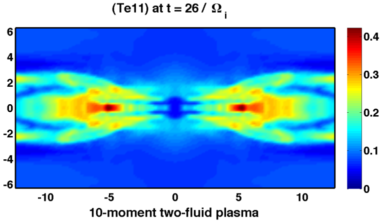


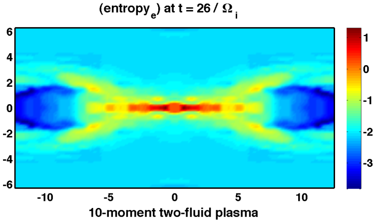
Electron gas at $t = 26$

Problem: the code crashes! Why? Look at electron gas dynamics:



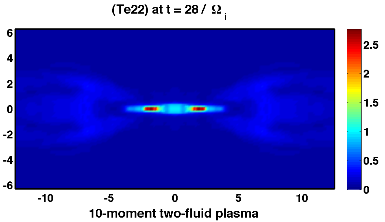


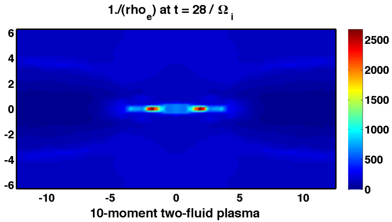


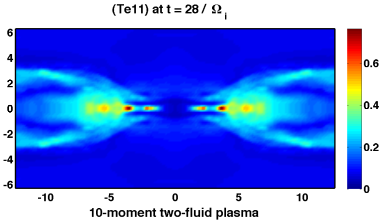


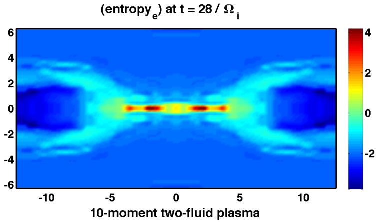
Electron gas at $t = 28$ (just before crashing)

Problem: the code crashes! Why? Look at electron gas dynamics:









Heating singularity ($\frac{m_\text{i}}{m_\text{e}} = 25$)

A heating singularity develops between 20% and 50% reconnected flux which crashes the code or produces a central magnetic island.

- $(\mathbb{T}_\text{e})_{yy}$ becomes large.
- $(\mathbb{T}_\text{e})_{xx}$ becomes small.
- ρ_e becomes small.
- electron entropy becomes large.

These difficulties prompted me to study whether nonsingular steady-state solutions exist for adiabatic plasma models.

Steady magnetic reconnection requires heat flux

Theorem. 2D rotationally symmetric steady magnetic reconnection must be singular in the vicinity of the X-point for an adiabatic model.

Argument. In a steady state solution that is symmetric under 180-degree rotation about the X-point, momentum evolution at the X-point says:

$$\text{rate of reconnection} = \mathbf{E}_3(0) = \frac{-\mathbf{R}_i}{en_i} + \frac{\nabla \cdot \mathbb{P}_i}{en_i}.$$

Assume a nonsingular steady solution. Then at the origin (0) no heat can be produced, so $\mathbf{R}_\mathrm{i} = 0$ at 0. Differentiating entropy evolution twice shows that $\nabla \cdot \mathbb{P}_\mathrm{i} = 0$ at 0. So there is no reconnection.

Part III: Heat flux closure

Heat flux closure

We need a nonzero heat flux closure. I advocate to use:

$$\mathbb{q}_{\rm s} = -\tfrac{2}{5} k_{\rm s} \widetilde{\boldsymbol{K}}_{\rm s} \vdots \operatorname{Sym3}\left(\frac{\mathbb{T}_{\rm s}}{T_{\rm s}} \cdot \nabla \mathbb{T}_{\rm s}\right);$$

here k is the heat conductivity. In the absence of a magnetic field $\widetilde{\boldsymbol{K}}$ is the identity tensor [McDonald and Groth, 2008].

What should $\widetilde{K}$ be in the presence of a magnetic field?

- depends on collision operator.
- I assume a BGK collision operator.

BGK collision operator

Recall the Boltzmann equation,

$$\partial_t f_s + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{a}_s f_s) = C_s.$$

The BGK collision operator relaxes f_s to a Maxwellian distribution:

$$C_s = \frac{f_{\mathcal{M}} - f_s}{\tau_s}.$$

Closure coefficients for heat flux tensor

For the heat flux coefficients a Chapman-Enskog expansion assuming a BGK collision operator yields

$$\begin{aligned}\widetilde{\boldsymbol{K}} = &\left(\mathbb{1}_{\parallel}^3 + \tfrac{3}{2}\mathbb{1}_{\parallel}(\mathbb{1}_{\perp}^2 + \mathbb{1}_{\wedge}^2)\right) \\ &+ \frac{3}{1+\varpi^2}\left(\mathbb{1}_{\perp}\mathbb{1}_{\parallel}^2 - \varpi\mathbb{1}_{\wedge}\mathbb{1}_{\parallel}^2\right) \\ &+ \frac{3}{1+4\varpi^2}\left(\frac{\mathbb{1}_{\perp}^2 - \mathbb{1}_{\wedge}^2}{2}\mathbb{1}_{\parallel} - 2\varpi\mathbb{1}_{\wedge}\mathbb{1}_{\perp}\mathbb{1}_{\parallel}\right) \\ &+ (k_0\mathbb{1}_{\perp}^3 + k_1\mathbb{1}_{\wedge}\mathbb{1}_{\perp}^2 + k_2\mathbb{1}_{\wedge}^2\mathbb{1}_{\perp} + k_3\mathbb{1}_{\wedge}^3);\end{aligned} \tag{1}$$

here $\varpi := \tau_{\mathrm{s}}\frac{q_{\mathrm{s}}}{m_{\mathrm{s}}}|\mathbf{B}|$ is gyrofrequency per collision frequency, $\mathbf{b} := \mathbf{B}/|\mathbf{B}|$ is the direction vector of the magnetic field, $\mathbb{1}$ is the identity matrix, and $\mathbb{1}_{\parallel} := \mathbf{bb}$, $\mathbb{1}_{\perp} := \mathbb{1} - \mathbb{1}_{\parallel}$, and $\mathbb{1}_{\wedge} = \mathbb{1} \times \mathbf{b}$ generate gyrotropic basis tensors. The remaining coefficients are...

Closure coefficients for heat flux tensor

$$
\begin{aligned}
k_3 :=& \frac{-6\varpi^3}{1+10\varpi^2+9\varpi^4} &&= -(2/3)\varpi^{-1} + \mathcal{O}(\varpi^{-3}),\\
k_2 :=& \frac{6\varpi^2+3\varpi(1+3\varpi^2)k_3}{1+7\varpi^2} &&= \mathcal{O}(\varpi^{-2}),\\
k_1 :=& \frac{-3\varpi+2\varpi k_2}{1+3\varpi^2} &&= -\varpi^{-1} + \mathcal{O}(\varpi^{-3}),\\
k_0 :=& 1+\varpi k_1 &&= \mathcal{O}(\varpi^{-2}).
\end{aligned}
$$

Closure coefficients for heat flux tensor

All tensor products in equation (1) are **splice symmetric products**, satisfying $2(AB)_{j_1 j_2 k_1 k_2} := A_{j_1 k_1} B_{j_2 k_2} + B_{j_1 k_1} A_{j_2 k_2}$ and

$$\begin{aligned} 3!(ABC)_{j_1 j_2 j_3 k_1 k_2 k_3} :=& A_{j_1 k_1} B_{j_2 k_2} C_{j_3 k_3} + A_{j_1 k_1} C_{j_2 k_2} B_{j_3 k_3} \\ &+ B_{j_1 k_1} A_{j_2 k_2} C_{j_3 k_3} + B_{j_1 k_1} C_{j_2 k_2} A_{j_3 k_3} \\ &+ C_{j_1 k_1} A_{j_2 k_2} B_{j_3 k_3} + C_{j_1 k_1} B_{j_2 k_2} A_{j_3 k_3} \end{aligned}$$

(so permute the letters and leave the indices unchanged).
For computational efficiency instead use **splice products**,

$$\begin{aligned} (AB)'_{j_1 j_2 k_1 k_2} &:= A_{j_1 k_1} B_{j_2 k_2}, \\ (ABC)'_{j_1 j_2 j_3 k_1 k_2 k_3} &:= A_{j_1 k_1} B_{j_2 k_2} C_{j_3 k_3}, \end{aligned}$$

and symmetrize at the end:

$$\mathbb{q}_\mathrm{s} = -\tfrac{2}{5} k_\mathrm{s}\, \mathrm{Sym}\left(\widetilde{\boldsymbol{K}}'_\mathrm{s} \,\vdots\, \mathrm{Sym3}\left(\frac{\mathbb{T}_\mathrm{s}}{T_\mathrm{s}} \cdot \nabla \mathbb{T}_\mathrm{s} \right) \right);$$

Future work

- Clean up my dissertation (!)
- Develop DoGPack.
- Develop Boundary Integral Positivity Limiter framework.
- Study 10-moment two-fluid tearing.
- Implement heat flux closure.
- Lorentz-invariant heat flux evolution.

Thanks!

I want to thank my committee, with special thanks to James Rossmanith for advising me, Jerry Brackbill for *lots* of help and advice, and Carl Sovinec for many consultations.

Also, I want to acknowledge people have been especially helpful to me in this research, particularly Ping Zhu, Nick Murphy, and Ammar Hakim.