Space Weather - Homework due Fr. Jan 11

The purpose of this assignment is to make sure that you understand the material from the lessons on November 29th and December 7th that you will be expected to know for the examination.

Please feel free to contact me if you have any questions. My contact information is:

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Problem 1

Complete the derivation of MHD that you started in Problem 3 of Homework 3:

1. Obtain conservation of *total energy density* by summing the energy (density) evolution equations of the ions and electrons. Again, assume that drift velocities $\mathbf{w}_\mathrm{s}$ are small and ignore them where appropriate. What would be the net contribution of collisions to total energy evolution if you were to retain resistive drag in the energy evolution equation in Problem 1?

2. Derive a **net current density** evolution equation by summing the current density evolution equations of the two species. Throw away all differentiated terms (with respect to space or time) and solve for the electric field to obtain **Ohm's law**. Assume charge neutrality and that the magnitude of the charge of the ions and the electrons is the same (e). (So $\rho_i = nm_i$ and $\rho_e = nm_e$, where n is the number density of either of the two species.) What simplification can you make in the case of a pair plasma? (For a pair plasma, $m_\mathrm{i} = m_\mathrm{e}$.)

Problem 2

Find the three main MHD waves by linearizing around a background state $(\rho_0, (\mathbf{u})_0 = \mathbf{0}, p_0, (\mathbf{B})_0$, where $(\mathbf{B})_0$ is parallel or perpendicular to the wave vector $\mathbf{k}$. You can linearize MHD and use the linearized equations as your starting point for each part.

1. Show that there are compressional acoustic waves (**sound waves**) that propagate parallel to the magnetic field independent

of the magnetic field strength.

2. Show that there are transverse magnetic field waves (**Alfvén waves**) that propagate parallel to the magnetic field.

3. Show that there are compressional magnetosonic waves (**fast magnetosonic waves**) that propagate perpendicular to the magnetic field.

Hint: assume that $\partial_y = 0 = \partial_z$. You can assume without loss of generality that $\mathbf{B} = |\mathbf{B}|\widehat{x}$ or $\mathbf{B} = |\mathbf{B}|\widehat{y}$. These assumption will put many zeros in the matrix of coefficients and will allow you to decouple it into small subsystems that you can easily solve. Alternatively, you can impose an ansatz (an assumption that helps you find a solution) and solve. The bottom line is that you have to exhibit a solution. *Remark: it is enough if you derive a wave equation for each of these problems with waves moving at the correct speed.*

Problem 3

Find the exact solution of the two-fluid system assuming that:

1. all spatial derivatives are zero and

2. the magnetic field is zero.

Hint: the system decouples into three separate 3x3 linear systems with constant coefficients, one for each dimension. They are all the same, so pick one of them and solve it. What frequency shows up in your solution? (Compare with slide number 3, titled "Modeling parameters", in the lecture presentation from November 29th.) *Remark: it is enough if you derive a scalar ODE with oscillations at the correct frequency.*

Problem 4

The momentum evolution equation that you derived in Problem 1 should have a $\mathbf{J} \times \mathbf{B}$ source term. Assume that $\partial_t \mathbf{E} = 0$ (another fundamental assumption of MHD). Use Ampere's law ($\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$), vector calculus, and the divergence constraint $\nabla \cdot \mathbf{B} = 0$ to write $\mathbf{J} \times \mathbf{B}$ as a divergence. This puts the total momentum evolution equation in conservation form, shows that momentum is conserved, and suggests a way to define the pressure of the magnetic field.

Problem 5 (Challenge problem)

The energy evolution equation that you derived in Problem 1 should have a $\mathbf{J} \cdot \mathbf{E}$ source term. Use vector calculus (ϵ_{ijk} may be handier) and the version of Maxwell's equations used by MHD to write it as a divergence plus the time derivative of something. (Hint: after doing Problems 4 and 5 you should have used each Maxwell equation used in MHD at least once.) Use this to put energy evolution in conservation form. How should the energy of the magnetic field be defined?