Fluid models from multi-fluid to resistive MHD

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Abstract: The fundamental plasma equations consist of Maxwell's equations for the electromagnetic field coupled to the kinetic equations for particle motion. The two-fluid model replaces the kinetic equations with fluid equations and is appropriate when intraspecies collisions are frequent enough to keep the distribution of particle velocities nearly symmetric. On time scales for which plasma oscillations are rapid, positive and negative charges must balance, and the plasma acts like a single, conducting fluid described by the equations of resistive magnetohydrodynamics (MHD).

Outline

1. Vlasov: fluid in phase space
2. Presentation of plasma models
3. Derivation of plasma models
4. MHD

Conservation law framework

Quantities:

- t = **time**
- $\mathbf{X}$ = **position**
- $U(t, \mathbf{X})$ = **balanced quantity**
- $\mathbf{F}(t, \mathbf{X})$ = **flux function** (e.g. $\mathbf{F}(U)$).
- $S(t, \mathbf{X}) = 0$ (no production of U).

Definitions:

- Ω = arbitrary region
- $\mathrm{d}\Omega$: volume element
- $\mathrm{d}t\,\mathrm{d}\Omega S$: production in volume element
- $\widehat{\mathbf{n}}$ = outward unit vector
- $\mathrm{d}\mathbf{A} = \widehat{\mathbf{n}}\mathrm{d}A$: surface element
- $\mathrm{d}t\,\mathrm{d}\mathbf{A} \cdot \mathbf{F}(t, \mathbf{X})$ = flux of U out of surface element. To see that flux is linear in $\mathrm{d}A$, consider that Ω can be approximated by a set of cells in a rectangular grid. $\mathrm{d}t\, dA_1 F_1$ gives flux across face perpendicular to first axis; dA_1 is area of projection of surface element onto first axis.
  Note: $F = T$ in picture.

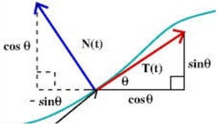
Balance law:

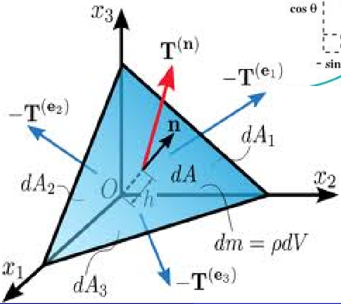
$$(\forall\Omega) \quad \int_\Omega U(t_1) - \int_\Omega U(t_0) = -\oint_{\partial\Omega} \mathrm{d}\mathbf{A} \cdot \int_{t_0}^{t_1} \mathbf{F}$$

$$\iff (\forall\Omega) \quad d_t \int_\Omega U = -\oint_{\partial\Omega} \mathrm{d}\mathbf{A} \cdot \mathbf{F}$$

$$\iff (\forall\Omega) \quad \int_\Omega (\partial_t U + \nabla \cdot \mathbf{F}) = 0$$

$$\iff \boxed{\partial_t U + \nabla \cdot \mathbf{F} = 0}.$$





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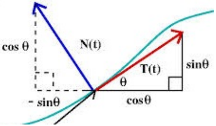
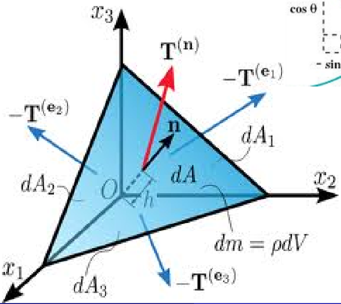
$$(\forall\Omega)\quad \int_\Omega U(t_1) - \int_\Omega U(t_0) = -\oint_{\partial\Omega}\mathrm{d}A\cdot\int_{t_0}^{t_1}\mathbf{F} + \int_\Omega\int_{t_0}^{t_1} S$$

$$\iff (\forall\Omega)\quad d_t\int_\Omega U = -\oint_{\partial\Omega}\mathrm{d}A\cdot\mathbf{F} + \int_\Omega S$$

$$\iff (\forall\Omega)\quad \int_\Omega(\partial_t U + \nabla\cdot\mathbf{F} - S) = 0$$

$$\iff \boxed{\partial_t U + \nabla\cdot\mathbf{F} = S}.$$





Transport Derivatives

Given:

- t = time
- $\mathbf{X}$ = position
- $\mathbf{V}(t, X)$ = velocity field
- $\alpha(t, \mathbf{x})$ = arbitrary function
- $\rho(t, \mathbf{x})$ = density convected by $\mathbf{V}$
- $\mathrm{d}_t := \frac{\mathrm{d}}{\mathrm{d}t}$
- $\boxed{\bar{\delta}_t \alpha := \partial_t \alpha + \nabla \cdot (\mathbf{V}\alpha)}$
  = **"transport derivative"** of α.
- $\boxed{\mathrm{d}_t \alpha := \partial_t \alpha + \mathbf{V} \cdot \nabla \alpha}$
  = material derivative of α.

Properties:

- $\boxed{\bar{\delta}_t \alpha = \mathrm{d}_t \alpha + \alpha \nabla \cdot \mathbf{V}}$.
- $\bar{\delta}_t(\alpha\beta) = \mathrm{d}_t(\alpha\beta) + (\nabla \cdot \mathbf{V})\alpha\beta$
  $= (\mathrm{d}_t\alpha)\beta + \alpha(\mathrm{d}_t\beta) + (\nabla \cdot \mathbf{V})\alpha\beta$
  $= (\bar{\delta}_t\alpha)\beta + \alpha(\mathrm{d}_t\beta)$.
- $\boxed{\bar{\delta}_t(\rho\beta) = \rho \mathrm{d}_t \beta}$.

Conservation of transported material:

$\rho(t, \mathbf{x})$ is transported by $\mathbf{V}$
$\iff \mathbf{F} := \mathbf{V}\rho$ is a flux for ρ
$\iff \boxed{\partial_t \rho + \nabla \cdot (\mathbf{V}\rho) = 0}$
$\iff \bar{\delta}_t \rho = 0$
$\iff d_t \rho + \rho \nabla \cdot \mathbf{V} = 0$
$\iff d_t \ln \rho = -\nabla \cdot \mathbf{V}$.

Incompressible flow:

$\mathbf{V}$ is incompressible
$\iff d_t \rho = 0$
$\iff d_t \ln \rho = 0$
$\iff \boxed{\nabla \cdot \mathbf{V} = 0}$
$\iff d_t \alpha = \bar{\delta}_t \alpha \quad (\forall \alpha)$.

Vlasov equation

Given:

- $\mathbf{x}$: position
- $\mathbf{v} = \dot{\mathbf{x}}$: velocity
- $\mathbf{a} = \dot{\mathbf{v}}$: acceleration
- $\tilde{f}_s$: number distribution of species s.
- $\tilde{f}_s(t, \mathbf{x}, \mathbf{v}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$: number of particles of species s in a region of state space with volume $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$.
- m_s: particle mass of species s
- q_s: particle charge of species s
- $f_s = m_s \tilde{f}_s$: mass distribution of species s.
- $\mathbf{a}_s = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$: Lorentz acceleration.
- $\mathbf{X} := (\mathbf{x}, \mathbf{v})$: position in state space.
- $\mathbf{V} := \dot{\mathbf{X}} = (\mathbf{v}, \mathbf{a}_s)$: velocity in state space.
- $(\mathbf{v} \times \mathbf{B})_i = \sum_j \sum_k \epsilon_{ijk} v_j B_k$ (cross product)
- ϵ_{ijk}: Levi-Civita symbol

We suppress the species index s when focusing on one species.

Theorem: Lorentz acceleration implies incompressible flow in phase space.

- Incompressible means $\nabla_{\mathbf{X}} \cdot \mathbf{V} = 0$.
- $\nabla_{\mathbf{X}} \cdot \mathbf{V} = \nabla_{\mathbf{x}} \cdot \mathbf{v} + \nabla_{\mathbf{v}} \cdot \mathbf{a}$
- $\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$ because $\mathbf{x}$ and $\mathbf{v}$ are independent variables.
- $\nabla_{\mathbf{v}} \cdot \mathbf{E}(t, \mathbf{x}) = 0$ for same reason.
- So $\nabla_{\mathbf{v}} \cdot \mathbf{a} = \frac{q}{m} \frac{\partial}{\partial \mathbf{v}_i} \epsilon_{ijk} \mathbf{v}_j B_k(t, \mathbf{x}) = 0$.

Vlasov equation (conservation of particles):

$f(t, \mathbf{X})$ is transported by $\mathbf{V}$

$$\iff \partial_t f + \nabla_{\mathbf{X}} \cdot (\mathbf{V} f) = 0$$

$$\iff \boxed{\boxed{\partial_t f + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f) + \nabla_{\mathbf{v}} \cdot (\mathbf{a} f) = 0}}$$

(conservation form)

$$\iff \boxed{\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{a} \cdot \nabla_{\mathbf{v}} \cdot f = 0}$$

$$\iff \partial_t f + \mathbf{V} \cdot \nabla_{\mathbf{X}} f = 0$$

Remark: conservation form is preferred for taking fluid moments.

Outline

Kinetic-Maxwell:

particle equations:

$$d_t \mathbf{x}_p = \mathbf{v}_p,$$
$$d_t \mathbf{v}_p = e \frac{q_p^\#}{m_p} (\mathbf{v}_p \times \mathbf{B}(\mathbf{x}_p) + \mathbf{E}(\mathbf{x}_p)) + \mathbf{r}$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$-c^{-2} \partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla \cdot \mathbf{B} = 0, \quad c^{-2} \nabla \cdot \mathbf{E} = \mu_0 \sigma.$$

charge-weighted moments:

$$\sigma(\mathbf{x}) := e \sum_p S_p(\mathbf{x} - \mathbf{x}_p) q_p^\#,$$
$$\mathbf{J}(\mathbf{x}) := e \sum_p S_p(\mathbf{x} - \mathbf{x}_p) q_p^\# \mathbf{v}_p.$$

Plugging $\dot{\mathbf{v}}_p = \frac{q_p}{m_p}(\mathbf{v}_p \times \mathbf{B} + \mathbf{E})$ into the time-derivative of mass ($\sum_p S_p m_p$), momentum ($\sum_p \mathbf{v}_p S_p m_p$), and energy ($\sum_p \frac{1}{2} v_p^2 S_p m_p$) density yields gas (i.e. fluid) equations.

Fluid approximation:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{(mass)},$$
$$\rho \mathrm{d}_t \mathbf{u} + \nabla p + \nabla \cdot \mathbb{P}^\circ = \mathbf{J} \times \mathbf{B} + \sigma \mathbf{E} + \mathbf{R} \qquad \text{(momentum)},$$
$$d_t p + \gamma p \nabla \cdot \mathbf{u} + \mathbb{P}^\circ : \nabla \mathbf{u} + \nabla \cdot \mathbf{q} = 0 \qquad \text{(energy)},$$

where we have used the definitions

$$\sigma := \sum qS, \quad \mathbf{J} := \sum \mathbf{v} qS, \quad p := \tfrac{1}{3} \sum c^2 mS,$$
$$\rho := \sum mS, \quad \rho \mathbf{u} := \sum \mathbf{v} mS, \quad \mathbb{P} := \sum \mathbf{cc} mS,$$
$$\mathbf{R} := \sum \mathbf{r} mS, \quad \mathbf{c} := \mathbf{v} - \mathbf{u}, \quad \mathbb{P}^\circ := \mathbb{P} - p\mathbb{I},$$

with the abbreviations

$$m := m_p, \quad S := S_p(\mathbf{x} - \mathbf{x}_p),$$
$$q := e q_p^\#, \quad \sum := \sum_p,$$

and the chain rule $\partial_t S = -\mathbf{v} \cdot \nabla S$.

Assuming that particle velocities for each species have a symmetric distribution implies $\mathbb{P}_s^\circ = 0$ and $\mathbf{q}_s = 0$, giving Euler gas dynamics for each species, hence the **ideal two-fluid Maxwell** plasma model.

Modeling parameters

Physical constants that define an ion-electron plasma:

1. e (charge of proton),
2. m_i, m_e (ion and electron mass),
3. c (speed of light),
4. μ_0 (vacuum permeability).

MHD parameters that characterize the state of a plasma:

1. n_0 (typical particle density),
2. T_0 (typical temperature),
3. B_0 (typical magnetic field).

Derived typical quantities:

- $p_0 := n_0 T_0$ (thermal pressure)
- $p_B := \frac{B_0^2}{2\mu_0}$ (magnetic pressure)
- $\rho_s := n_0 m_s$ (mass density).

Collision periods:

- τ_s: period of relaxation of species s toward Maxwellian

Collisionless time, velocity, and space scale parameters:

plasma frequencies: $\omega_{p,s}^2 := \frac{\mu_0 n_0 (ce)^2}{m_s}$,

gyrofrequencies: $\omega_{g,s} := \frac{eB_0}{m_s}$,

thermal velocities: $v_{t,s}^2 := \frac{2p_0}{\rho_s}$,

Alfvén speeds: $v_{A,s}^2 := \frac{2p_B}{\rho_s} = \frac{B_0^2}{\mu_0 m_s n_0}$,

Debye length: $\lambda_D := \frac{v_{t,s}}{\omega_{p,s}} = \sqrt{\frac{T_0}{n_0\mu_0(ce)^2}}$,

gyroradii: $r_{g,s} := \frac{v_{t,s}}{\omega_{g,s}} = \frac{m_s v_{t,s}}{eB_0}$,

skin depths: $\delta_s := \frac{v_{A,s}}{\omega_{g,s}} = \frac{c}{\omega_{p,s}} = \sqrt{\frac{m_s}{\mu_0 n_s e^2}}$.

plasma $\beta := \frac{p_0}{p_B} = \left(\frac{v_{t,s}}{v_{A,s}}\right)^2 = \left(\frac{r_{g,s}}{\delta_s}\right)^2$.

non-MHD ratio: $\frac{c}{v_{A,s}} = \frac{r_{g,s}}{\lambda_D} = \frac{\omega_{p,s}}{\omega_{g,s}}$.

1. **kinetic-Maxwell**

   ↓ fast collisions (${\tau_s}^{-1} \to \infty$)

2. **ideal two-fluid Maxwell**: Euler gas for each species: ρ_s, $\mathbf{u}_s$, p_s

   ↓ fast oscillations ($e \to \infty$)

3. **relativistic ideal MHD**: perfectly conducting gas

   ↓ fast light waves ($c \to \infty$)

4. **classical ideal MHD**: perfectly conducting gas: $\mathbf{E} = \mathbf{B} \times \mathbf{u}$.

Two-fluid Maxwell:

gas evolution:

$$\partial_t \rho_s + \nabla \cdot (\rho_s \mathbf{u}_s) = 0,$$
$$\rho_s d_t^s \mathbf{u}_s + \nabla p_s = \mathbf{J}_s \times \mathbf{B} + \sigma_s \mathbf{E} + \mathbf{R}_s,$$
$$d_t^s p_s + \gamma p_s \nabla \cdot \mathbf{u}_s = \frac{2}{3}\frac{m_{red}}{m_s} Q$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$-c^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla \cdot \mathbf{B} = 0, \quad c^{-2}\nabla \cdot \mathbf{E} = \mu_0 \sigma,$$
$$\mathbf{J} := \mathbf{J}_i + \mathbf{J}_e, \quad \mathbf{J}_s := \sigma_s \mathbf{u}_s,$$
$$\sigma := \sigma_i + \sigma_e, \quad \sigma_s := \pm \frac{e}{m_s}\rho_s.$$

closure:

$$-\mathbf{R}_i = \mathbf{R}_e = e^2 n_e n_i \boldsymbol{\eta} \cdot (\mathbf{u}_i - \mathbf{u}_e)$$
$$\approx en\boldsymbol{\eta} \cdot \mathbf{J},$$
$$Q = -\textstyle\sum_s \mathbf{R}_s \cdot \mathbf{u}_s \approx \mathbf{J} \cdot \boldsymbol{\eta} \cdot \mathbf{J}.$$

Quasi-relativistic MHD ($e \to \infty$):

gas evolution:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{(mass)},$$
$$\rho \mathrm{d}_t \mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B} + \sigma \mathbf{E} \qquad \text{(momentum)},$$
$$d_t p + \gamma p \nabla \cdot \mathbf{u} = \frac{2}{3} \mathbf{J} \cdot \boldsymbol{\eta} \cdot \mathbf{J} \qquad \text{(thermal energy)}.$$

magnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \qquad \text{(magnetic field)},$$
$$\mathbf{E} = \mathbf{B} \times \mathbf{u} + \boldsymbol{\eta} \cdot \mathbf{J} \qquad \text{(Ohm's law)},$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(divergence constraint)},$$
$$\mu_0 \mathbf{J} := \nabla \times \mathbf{B} - c^{-2}\partial_t \mathbf{E} \qquad \text{(Ampere's law for current)},$$
$$\mu_0 \sigma := c^{-2} \nabla \cdot \mathbf{E} \qquad \text{(quasineutrality)}.$$

definitions:

$$d_t^s := \partial_t + \mathbf{u}_s \cdot \nabla, \qquad \gamma := \frac{5}{3},$$
$$d_t := \partial_t + \mathbf{u} \cdot \nabla, \qquad m_{red}^{-1} := \sum_s m_s^{-1}.$$

Two-fluid Maxwell:

gas evolution:

$$\partial_t \rho_s + \nabla\cdot(\rho_s \mathbf{u}_s) = 0,$$
$$\rho_s d_t^s \mathbf{u}_s + \nabla p_s = \mathbf{J}_s \times \mathbf{B} + \sigma_s \mathbf{E} + \mathbf{R}_s,$$
$$d_t^s p_s + \gamma p_s \nabla\cdot\mathbf{u}_s = \frac{2}{3}\frac{m_{\rm red}}{m_s} Q$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0,$$
$$-c^{-2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla\cdot\mathbf{B} = 0, \quad c^{-2}\nabla\cdot\mathbf{E} = \mu_0\sigma,$$
$$\mathbf{J} := \mathbf{J}_i + \mathbf{J}_e, \quad \mathbf{J}_s := \sigma_s \mathbf{u}_s,$$
$$\sigma := \sigma_i + \sigma_e, \quad \sigma_s := \pm\frac{e}{m_s}\rho_s.$$

closure:

$$-\mathbf{R}_i = \mathbf{R}_e = e^2 n_e n_i \boldsymbol{\eta}\cdot(\mathbf{u}_i - \mathbf{u}_e)$$
$$\approx en\boldsymbol{\eta}\cdot\mathbf{J},$$
$$Q = -\sum_s \mathbf{R}_s\cdot\mathbf{u}_s \approx \mathbf{J}\cdot\boldsymbol{\eta}\cdot\mathbf{J}.$$

Classical MHD ($e\to\infty$, $c\to\infty$):

gas evolution:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad \text{(mass)},$$
$$\rho d_t \mathbf{u} + \nabla p = \mathbf{J}\times\mathbf{B} \qquad \text{(momentum)},$$
$$d_t p + \gamma p \nabla\cdot\mathbf{u} = \frac{2}{3}\mathbf{J}\cdot\boldsymbol{\eta}\cdot\mathbf{J} \qquad \text{(thermal energy)}.$$

magnetic field:

$$\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0 \qquad \text{(magnetic field)},$$
$$\mathbf{E} = \mathbf{B}\times\mathbf{u} + \boldsymbol{\eta}\cdot\mathbf{J} \qquad \text{(Ohm's law)},$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \text{(divergence constraint)},$$
$$\mu_0\mathbf{J} := \nabla\times\mathbf{B} \qquad \text{(Ampere's law for current)},$$
$$\mu_0\sigma := 0 \qquad \text{(neutrality)}.$$

definitions:

$$d_t^s := \partial_t + \mathbf{u}_s\cdot\nabla, \qquad \gamma := \frac{5}{3},$$
$$d_t := \partial_t + \mathbf{u}\cdot\nabla, \qquad m_{\rm red}^{-1} := \sum_s m_s^{-1}.$$

Outline

kinetic-Maxwell (the "truth")

particle evolution:

$$d_t \mathbf{x}_p = \mathbf{v}_p,$$
$$d_t \mathbf{v}_p = \mathbf{a}_\mathrm{p}(\mathbf{x}_p, \mathbf{v}_p),$$
$$\mathbf{a}_\mathrm{p} = \frac{q_\mathrm{p}}{m_\mathrm{p}} (\mathbf{v}_\mathrm{p} \times \mathbf{B}(\mathbf{x}_\mathrm{p}) + \mathbf{E}(\mathbf{x}_\mathrm{p})) + \mathbf{r}_\mathrm{p}.$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$-c^{-2} \partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla \cdot \mathbf{B} = 0, \quad c^{-2} \nabla \cdot \mathbf{E} = \mu_0 \sigma.$$

charge-weighted moments:

$$\sigma(\mathbf{x}) := \sum_p S_p(\mathbf{x}) \mathbf{q}_p,$$
$$\mathbf{J}(\mathbf{x}) := \sum_p S_p(\mathbf{x}) \mathbf{q}_p \mathbf{v}_p;$$

here $S_\mathrm{p}(\mathbf{x}) = S(\mathbf{x} - \mathbf{x}_\mathrm{p})$ is the shape function of particle p, $\mathbf{x}_\mathrm{p}$ is its position, $\mathbf{v}_\mathrm{p}$ is its velocity, $\mathbf{r}_\mathrm{p}$ is collisional drag, $\mathbf{E}$ is electric field, $\mathbf{B}$ is magnetic field, $\mathbf{J}$ is current, and σ is charge density.

Collisional drag.

The term $\mathbf{r}_\mathrm{p}$ can be used to incorporate gravitational acceleration, but in this context we introduce $\mathbf{r}_\mathrm{p}$ to account for microscale interactions not accounted for by macroscale smoothed versions of the electromagnetic field.

Collisional drag must conserve momentum and energy:

$$\begin{aligned} \sum \mathbf{r}_\mathrm{p} m_\mathrm{p} S_\mathrm{p} &= 0 \qquad \text{(momentum)}, \\ \sum \mathbf{r}_\mathrm{p} \cdot \mathbf{v}_\mathrm{p} m_\mathrm{p} S_\mathrm{p} &= 0 \qquad \text{(energy)}. \end{aligned} \tag{1}$$

Collision operator [aside].

For each species s, specifying $\mathbf{r}$ is equivalent to specifying a collision operator $\mathcal{C}$. Indeed, requiring the collisional Vlasov equation

$$\partial_t f + \nabla \cdot (\mathbf{v} f) + \nabla_\mathbf{v} \cdot (\mathbf{a} f) = \mathcal{C}$$

to agree with the "drag force" Vlasov equation

$$\partial_t f + \nabla \cdot (\mathbf{v} f) + \nabla_\mathbf{v} \cdot ((\mathbf{a} + \mathbf{r}) f) = 0$$

reveals that

$$-\nabla_\mathbf{v} \cdot (\mathbf{r} f) = \mathcal{C}$$

must hold; to solve, set $\mathbf{r} f = \nabla \phi$, where $-\nabla_\mathbf{v}^2 \phi = \mathcal{C}$.

particle evolution:

$$d_t \mathbf{x}_p = \mathbf{v}_p,$$
$$d_t \mathbf{v}_p = \mathbf{a}_{\mathrm{p}}(\mathbf{x}_p, \mathbf{v}_p),$$
$$\mathbf{a}_{\mathrm{p}} = \frac{q_{\mathrm{p}}}{m_{\mathrm{p}}}\left(\mathbf{v}_{\mathrm{p}} \times \mathbf{B}(\mathbf{x}_{\mathrm{p}}) + \mathbf{E}(\mathbf{x}_{\mathrm{p}})\right) + \mathbf{r}_{\mathrm{p}}. \quad (2)$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$-c^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla \cdot \mathbf{B} = 0, \quad c^{-2}\nabla \cdot \mathbf{E} = \mu_0 \sigma.$$

charge-weighted moments:

$$\sigma(\mathbf{x}) := \sum_p S_p(\mathbf{x})\mathbf{q}_p,$$
$$\mathbf{J}(\mathbf{x}) := \sum_p S_p(\mathbf{x})\mathbf{q}_p \mathbf{v}_p;$$

here $S_{\mathrm{p}}(\mathbf{x}) = S(\mathbf{x} - \mathbf{x}_{\mathrm{p}})$ is the shape function of particle p, $\mathbf{x}_{\mathrm{p}}$ is its position, $\mathbf{v}_{\mathrm{p}}$ is its velocity, $\mathbf{r}_{\mathrm{p}}$ is collisional drag, $\mathbf{E}$ is electric field, $\mathbf{B}$ is magnetic field, $\mathbf{J}$ is current, and σ is charge density.

Fluid models evolve mass-weighted moments:

$$\rho(\mathbf{x}) := \sum_{\mathrm{p}} m_{\mathrm{p}} S_{\mathrm{p}}(\mathbf{x}) \qquad \text{(mass)},$$
$$\mathbf{M}(\mathbf{x}) := \sum_{\mathrm{p}} \mathbf{v}_{\mathrm{p}} m_{\mathrm{p}} S_{\mathrm{p}}(\mathbf{x}) \qquad \text{(momentum)},$$
$$\mathcal{E}(\mathbf{x}) := \sum_{\mathrm{p}} \frac{1}{2}|\mathbf{v}_{\mathrm{p}}|^2 m_{\mathrm{p}} S_{\mathrm{p}}(\mathbf{x}) \qquad \text{(energy)},$$

To abbreviate we drop the particle summation index p and the independent variable $\mathbf{x}$ and write

$$\sigma := \sum qS \qquad \text{(charge)},$$
$$\rho := \sum mS \qquad \text{(mass)},$$
$$\mathbf{J} := \sum \mathbf{v}qS \qquad \text{(current)},$$
$$\mathbf{M} := \sum \mathbf{v}mS \qquad \text{(momentum)},$$
$$\mathcal{E} := \sum \frac{1}{2}|v|^2 mS \qquad \text{(energy)}.$$

To get fluid equations, differentiate and use:

- $\boxed{\dot{\mathbf{v}} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{r}}$
- $\partial_t S(\mathbf{x} - \mathbf{x}_{\mathrm{p}}(t)) = -\dot{\mathbf{v}}_{\mathrm{p}} \cdot \nabla S(\mathbf{x} - \mathbf{x}_{\mathrm{p}})$, i.e., $\boxed{\partial_t S = -\mathbf{v} \cdot \nabla S}$.

Moment evolution: from kinetic to fluid

Given definitions:

- $\chi(\mathbf{v}) = \begin{cases} 1 & \text{zeroth moment} \\ \mathbf{v} & \text{first moment} \\ v^2 & \text{second moment} \end{cases}$
- $\langle\chi\rangle := \frac{\sum \chi m S}{\sum m S}$ (**statistical mean** of χ).
- $\rho := \sum m S$ (mass density)
- $\boxed{\rho\langle\chi\rangle := \sum \chi m S}$. (generic moment)
- $\mathbf{u} := \langle\mathbf{v}\rangle$ (fluid velocity)
- $\mathbf{c} := \mathbf{v} - \mathbf{u}$ (thermal velocity)
- $\bar{\delta}_t\alpha := \partial_t\alpha + \nabla\cdot(\mathbf{u}\alpha)$ ("transport derivative").
- $d_t\alpha := \partial_t\alpha + \mathbf{u}\cdot\nabla\alpha$. (advective derivative).
- $\mathbf{M} = \rho\mathbf{u}$ (momentum).
- subscript s restricts sums to particles of species s.
- $n_\mathrm{s} = \sum_{\mathrm{p}\in\mathrm{s}} S_\mathrm{p} = \frac{1}{m_\mathrm{s}}\rho_\mathrm{s}$ (number density)

Generic mass moment evolution:

$$\partial_t \sum \chi m S = \sum \chi m \partial_t S + \sum \dot{\chi} m S$$

$$\iff \partial_t(\rho\langle\chi\rangle) + \sum \chi\mathbf{v}\cdot m\nabla S = \sum \dot{\chi} m S$$

$$\iff \partial_t(\rho\langle\chi\rangle) + \nabla\cdot\sum \mathbf{v}\chi m S = \sum \dot{\mathbf{v}}\cdot\nabla_\mathbf{v}\chi m S$$

$$\iff \partial_t(\rho\langle\chi\rangle) + \nabla\cdot(\rho\langle\mathbf{v}\chi\rangle) = \rho\langle\dot{\mathbf{v}}\cdot\nabla_\mathbf{v}\chi\rangle$$

$$\iff \boxed{\bar{\delta}_t(\rho\langle\chi\rangle) + \nabla\cdot(\rho\langle\mathbf{c}\chi\rangle) = \rho\langle\mathbf{a}\cdot\nabla_\mathbf{v}\chi\rangle}; \qquad (3)$$

in the last step we have used that $\langle\mathbf{v}\chi\rangle = \langle(\mathbf{u}+\mathbf{c})\chi\rangle = \mathbf{u}\langle\chi\rangle + \langle\mathbf{c}\chi\rangle$ and $\bar{\delta}_t(\rho\langle\chi\rangle) = \partial_t(\rho\langle\chi\rangle) + \nabla\cdot(\mathbf{u}\langle\chi\rangle)$.

Mass continuity. ($\chi = 1$).

If $\chi = 1$, then $\langle\mathbf{c}\chi\rangle = \langle\mathbf{c}\rangle = 0$ and $\nabla_\mathbf{v}\chi = 0$, so we simply get $\bar{\delta}_t\rho = 0$, that is,

$$\partial_t\rho + \nabla\cdot(\mathbf{u}\rho) = 0.$$

Exercise: Using mass continuity, show that $\bar{\delta}_t(\rho\langle\chi\rangle) = \rho d_t\langle\chi\rangle$.

Charge continuity.

Differentiating the definition of charge density gives $\partial_t\sigma = \partial_t\sum qS = -\sum \mathbf{v}\cdot q\nabla S = -\nabla\cdot\sum \mathbf{v}qS$, i.e., the flux of charge is the current:

$$\boxed{\partial_t\sigma + \nabla\cdot\mathbf{J} = 0.}$$

Mass continuity (again).

Replacing q with m in charge density evolution shows that mass flux coincides with the (classical) definition of momentum:

$$\boxed{\partial_t\rho + \nabla\cdot(\mathbf{u}\rho) = 0}, \qquad (4)$$

that is (restricting to species s), $\partial_t\rho_\mathrm{s} + \nabla\cdot(\mathbf{u}_\mathrm{s}\rho_\mathrm{s}) = 0$; dividing by m_s gives continuity of number density $n_\mathrm{s} = \rho_\mathrm{s}/m_\mathrm{s}$ for species s:

$$\boxed{\partial_t n_\mathrm{s} + \nabla\cdot(\mathbf{u}_\mathrm{s} n_\mathrm{s}) = 0}.$$

Vlasov equation [aside].

The Vlasov equation is simply the continuity equation in six-dimensional phase-space. To see this:

- Use $\mathbf{X} = (\mathbf{x}, \mathbf{v})$ to denote a point in phase space.
- Use $\mathbf{V} = \dot{\mathbf{X}} = (\mathbf{v}, \mathbf{a})$ to denote velocity in phase space.
- Write the particle distribution function (for a species of particles) as the sum of particle shape functions: $f = \sum_\mathrm{p} \mathcal{S}_\mathrm{p}$.
- Observe that $\mathbf{V}(\mathbf{X})$, i.e. the fluid is “cold.”
- Assume that the shape of a particle in phase space is a delta function (unit spike): $\mathcal{S}_\mathrm{p}(\mathbf{X}) = \delta(\mathbf{X} - \mathbf{X}_\mathrm{p})$.

Then the continuity equation $\partial_t n_\mathrm{s} + \nabla\cdot(\mathbf{u}_\mathrm{s} n_\mathrm{s}) = 0$ becomes the Vlasov equation $\partial_t f_\mathrm{s} + \nabla_\mathbf{X}\cdot(\mathbf{V}_\mathrm{s} f_\mathrm{s}) = 0$, i.e.,

$$\boxed{\partial_t f + \nabla_\mathbf{x}\cdot(\mathbf{v}f) + \nabla_\mathbf{v}\cdot(\mathbf{a}f)}.$$

In gory detail:

$$
\begin{aligned}
-\partial_t f &= -\textstyle\sum_\mathrm{p}\partial_t\mathcal{S}_\mathrm{p}\\
&= \textstyle\sum_\mathrm{p}\mathbf{V}_\mathrm{p}\cdot\nabla_\mathbf{X}\mathcal{S}_\mathrm{p}\\
&= \nabla_\mathbf{X}\cdot\textstyle\sum_\mathrm{p}\mathbf{V}_\mathrm{p}\mathcal{S}_\mathrm{p}\\
&= \nabla_\mathbf{X}\cdot\textstyle\sum_\mathrm{p}\mathbf{V}(\mathbf{X}_\mathrm{p})\delta(\mathbf{X}-\mathbf{X}_p)\\
&= \nabla_\mathbf{X}\cdot\textstyle\sum_\mathrm{p}\mathbf{V}(\mathbf{X})\delta(\mathbf{X}-\mathbf{X}_p)\\
&= \nabla_\mathbf{X}\cdot\left(\mathbf{V}(\mathbf{X})\textstyle\sum_\mathrm{p}\delta(\mathbf{X}-\mathbf{X}_p)\right)\\
&= \nabla_\mathbf{X}\cdot(\mathbf{V}(\mathbf{X})f)\\
&= \nabla_\mathbf{x}\cdot(\mathbf{v}f) + \nabla_\mathbf{v}\cdot(\mathbf{a}f)\\
&= \mathbf{v}\cdot\nabla_\mathbf{x} f + \mathbf{a}\cdot\nabla_\mathbf{v} f,
\end{aligned}
$$

where the last step follows from the incompressibility condition $\nabla_\mathbf{v}\cdot\mathbf{a} = 0$.

Taking moments: momentum density evolution

Given definitions:

- $\mathbf{u} := \langle \mathbf{v} \rangle$ (bulk velocity)
- $\mathbf{c} := \mathbf{v} - \mathbf{u}$ (thermal velocity)
- $\mathbf{M} := \rho \mathbf{u}$ (momentum)
- $\mathbf{R} := \sum \mathbf{r} m S$ (collisional drag)
- $\mathbb{P} := \rho \langle \mathbf{cc} \rangle$ (pressure tensor)
- $p := \frac{1}{3} \rho \langle |\mathbf{c}|^2 \rangle$ (pressure)
- $\mathbb{P}^\circ := \mathbb{P} - p\,\mathbb{I}$ (deviatoric pressure)
- $\bar{\delta}_t^{\mathrm{s}} \alpha := \partial_t \alpha + \nabla \cdot (\mathbf{u}_{\mathrm{s}} \alpha)$ ("transport derivative" for $\mathbf{u}_{\mathrm{s}}$).

Remarks:

- If restricting to species s, then denote quantities as $\mathbf{u}_{\mathrm{s}}$, $\mathbf{R}_{\mathrm{s}}$, etc.
- Including all particles, the drag force cancels: $\mathbf{R} = \sum_{\mathrm{s}} \mathbf{R}_{\mathrm{s}} = 0$.
- $\mathbb{P}^\circ = 0$ if the distribution of particle velocities is *isotropic* (the same in all directions).

Momentum balance ($\chi = \mathbf{v}$):

- Recall generic moment evolution (Eqn. (3)):

$$\bar{\delta}_t(\langle \rho \chi \rangle) + \nabla_{\mathbf{x}} \cdot (\rho \langle \mathbf{c} \chi \rangle) = \rho \langle \mathbf{a} \cdot \nabla_{\mathbf{v}} \chi \rangle$$

- Observe that $\langle \mathbf{c} \rangle = 0$ (since $\mathbf{c} = \mathbf{v} - \mathbf{u}$ and $\langle \mathbf{v} \rangle = \mathbf{u}$). So $\langle \mathbf{vv} \rangle = \langle (\mathbf{u} + \mathbf{c})(\mathbf{u} + \mathbf{c}) \rangle = \mathbf{uu} + \mathbf{u}\langle \mathbf{c} \rangle + \langle \mathbf{c} \rangle \mathbf{u} + \langle \mathbf{cc} \rangle$. That is, $\langle \mathbf{vv} \rangle = \mathbf{uu} + \mathbb{P}$. Thus, since $\nabla_{\mathbf{v}} \cdot \mathbf{v} = \mathbb{I}$,

$$\bar{\delta}_t(\rho \mathbf{u}) + \nabla \cdot \mathbb{P} = \rho \langle \mathbf{a} \rangle.$$

- But $\langle \mathbf{a} \rangle = \frac{q}{m}(\mathbf{E} + \mathbf{u} \times \mathbf{B})$. Thus:

$$\boxed{\bar{\delta}_t(\rho \mathbf{u}) + \nabla \cdot \mathbb{P} = \sigma \mathbf{E} + \mathbf{J} \times \mathbf{B} + \mathbf{R}}. \qquad (5)$$

- **Kinetic energy balance** for species s equals momentum balance dot $\mathbf{u}$:

$$\bar{\delta}_t^{\mathrm{s}}(\rho_{\mathrm{s}} \tfrac{1}{2} |\mathbf{u}_{\mathrm{s}}|^2) + \mathbf{u}_{\mathrm{s}} \cdot (\nabla \cdot \mathbb{P}_{\mathrm{s}}) = \mathbf{J}_{\mathrm{s}} \cdot \mathbf{E} + \mathbf{R}_{\mathrm{s}} \cdot \mathbf{u}_{\mathrm{s}}$$

Taking moments: energy

Given definitions:

- $\mathcal{E} := \rho\langle \frac{1}{2} v^2\rangle$ (energy density)
- $\mathbb{P} := \rho\langle \mathbf{cc}\rangle$ (pressure tensor)
- $\mathbf{q} := \rho\langle \frac{1}{2}\mathbf{c}|\mathbf{c}|^2\rangle$ (heat flux)
- $Q := \sum \mathbf{r}\cdot\mathbf{c}$ (collisional heating)

Relationships:

- energy = kinetic plus thermal: $\langle|\mathbf{v}|^2\rangle = |\mathbf{u}|^2 + \langle|\mathbf{c}|^2\rangle$, i.e., $\rho\langle\frac{1}{2}|\mathbf{v}|^2\rangle = \rho\frac{1}{2}|\mathbf{u}|^2 + \rho\langle\frac{1}{2}|\mathbf{c}|^2\rangle$.
- pressure is $\frac{2}{3}$ the thermal energy: $p := \frac{1}{3}\rho\langle|\mathbf{c}|^2\rangle$, so $\mathcal{E} = \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{3}{2}p$.

Remarks:

- If restricting to species s, write e.g. Q_s.
- Including all particles, collisional energy production cancels: $\sum \mathbf{r}\cdot\mathbf{v}mS = 0$, i.e., $\mathbf{R}\cdot\mathbf{u} + Q = \sum_\mathrm{s}(\mathbf{R}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s} + Q_\mathrm{s}) = 0$
- $\mathbf{q} = 0$ if the distribution of particle velocities is symmetric.

Energy balance ($\chi = \frac{1}{2}|\mathbf{v}|^2$):

- Recall generic moment evolution (Eqn. (3)):

$$\rho \mathrm{d}_t\langle\chi\rangle + \nabla_\mathbf{x}\cdot(\rho\langle\mathbf{c}\chi\rangle) = \rho\langle\mathbf{a}\cdot\nabla_\mathbf{v}\chi\rangle$$

- For $\chi = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$, using that:
  - $\rho\langle\frac{1}{2}\mathbf{cv}\cdot\mathbf{v}\rangle = \rho\langle\mathbf{cc}\rangle\cdot\mathbf{u} + \rho\langle\frac{1}{2}\mathbf{cc}\cdot\mathbf{c}\rangle = \mathbb{P}\cdot\mathbf{u} + \mathbf{q}$,
  - $\rho\langle\mathbf{a}\cdot\mathbf{v}\rangle = \rho\langle\frac{q}{m}\mathbf{E}\cdot\mathbf{v}\rangle = \mathbf{E}\cdot\frac{q}{m}\rho\mathbf{u} = \mathbf{E}\cdot\mathbf{J}$ (that is, $\langle\mathbf{a}\cdot\mathbf{v}\rangle = \langle\mathbf{a}\rangle\cdot\langle\mathbf{v}\rangle$), and
  - $\sum\mathbf{r}\cdot\mathbf{v}mS = \sum\mathbf{r}\cdot\mathbf{u}mS + \sum\mathbf{r}\cdot\mathbf{c}mS = \mathbf{R}\cdot\mathbf{u} + Q$,

$$\boxed{\bar{\delta}_t\mathcal{E} + \nabla\cdot(\mathbb{P}\cdot\mathbf{u} + \mathbf{q}) = \mathbf{J}\cdot\mathbf{E} + \mathbf{R}\cdot\mathbf{u} + Q} \quad (6)$$

Thermal energy balance for species s:

- Recall kinetic energy balance:

$$\bar{\delta}_t^\mathrm{s}(\rho_\mathrm{s}\frac{1}{2}|\mathbf{u}_\mathrm{s}|^2) + \mathbf{u}_\mathrm{s}\cdot(\nabla\cdot\mathbb{P}_\mathrm{s}) = \mathbf{J}_\mathrm{s}\cdot\mathbf{E} + \mathbf{R}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}$$

- Thermal energy balance equals energy balance minus kinetic energy balance:

$$\bar{\delta}_t^\mathrm{s}(\rho_\mathrm{s}\langle\frac{1}{2}|\mathbf{c}_\mathrm{s}|^2\rangle) + \mathbb{P}_\mathrm{s}:\nabla\mathbf{u}_\mathrm{s} + \nabla\cdot\mathbf{q}_\mathrm{s} = Q_\mathrm{s}$$

Full fluid equations (one species):

Gathering together equations (4), (5), and (6) and restricting to species s, we have a system of balance laws for the mass(1) + momentum(3) + energy(1) = 5 conserved moments:

$$\begin{aligned} \bar{\delta}_t^{\mathrm{s}}\rho_{\mathrm{s}} &= 0 \\ \bar{\delta}_t^{\mathrm{s}}(\rho_{\mathrm{s}}\mathbf{u}_{\mathrm{s}}) + \nabla\cdot\mathbb{P}_{\mathrm{s}} &= \sigma_{\mathrm{s}}\mathbf{E} + \mathbf{J}_{\mathrm{s}}\times\mathbf{B} + \mathbf{R}_{\mathrm{s}} \\ \bar{\delta}_t^{\mathrm{s}}\mathcal{E}_{\mathrm{s}} \quad + \nabla\cdot(\mathbb{P}_{\mathrm{s}}\cdot\mathbf{u}_{\mathrm{s}} + \mathbf{q}_{\mathrm{s}}) &= \mathbf{J}_{\mathrm{s}}\cdot\mathbf{E} + \mathbf{R}_{\mathrm{s}}\cdot\mathbf{u}_{\mathrm{s}} + Q_{\mathrm{s}} \end{aligned} \tag{7}$$

MHD fluid equations:

The bulk fluid quantities of MHD are defined by

$$\begin{aligned} \rho &:= \rho_{\mathrm{i}} \quad + \rho_{\mathrm{e}}, \\ \rho\mathbf{u} &:= \rho_{\mathrm{i}}\mathbf{u}_{\mathrm{i}} + \rho_{\mathrm{e}}\mathbf{u}_{\mathrm{e}}, \\ \mathcal{E} &:= \mathcal{E}_{\mathrm{i}} \quad + \mathcal{E}_{\mathrm{e}}. \end{aligned}$$

One-fluid MHD assumes that the fluid velocity is the same for all species: $\mathbf{u}_i \approx \mathbf{u}_{\mathrm{e}}$. In this case, summing each equation in System (7) over ions ($\mathrm{s} = \mathrm{i}$) and electrons ($\mathrm{s} = \mathrm{e}$) gives the MHD equations, which are exactly the same but without the subscript s. The interspecies collision terms involving $\mathbf{R}_{\mathrm{s}}$ and Q_{s} cancel and disappear by the conservation laws (1).

Remarks

- System (7) is in the form

$$\begin{aligned} \bar{\delta}_t U + \nabla\cdot\widetilde{\mathbf{F}} = S, \quad \text{i.e.,} \\ \partial_t U + \nabla\cdot(\mathbf{u}U + \widetilde{\mathbf{F}}) = S, \end{aligned}$$

  which is in the balance form

$$\partial_t U + \nabla\cdot\mathbf{F} = S.$$

- One-fluid MHD assumes $\mathbf{u}_{\mathrm{i}} \approx \mathbf{u}_{\mathrm{e}}$, which holds in the limit $e \to \infty$. To see this, look at the charge density $\sigma = e(n_{\mathrm{i}} - n_{\mathrm{e}})$ and current density $\mathbf{J} = e(\mathbf{u}_{\mathrm{i}}n_{\mathrm{i}} - \mathbf{u}_{\mathrm{e}}n_{\mathrm{e}})$. As $e \to \infty$, $\mathbf{J}$ and σ approach finite limiting values (because $\mu_0\sigma = c^{-2}\nabla\cdot\mathbf{E}$ and $\mu_0\mathbf{J} = \nabla\times\mathbf{B} - c^{-2}\partial_t\mathbf{E}$). Since $\sigma/e \to 0$ and $\mathbf{J}/e \to 0$, in the limit $e \to \infty$, $n_{\mathrm{i}} = n_{\mathrm{e}}$ and thus $\mathbf{u}_{\mathrm{i}} = \mathbf{u}_{\mathrm{e}}$.

The pressure tensor is usually separated out into its scalar part $p_s = \frac{1}{3}\operatorname{tr}\mathbb{P}_s$ (where $\operatorname{tr}\mathbb{P} := \mathbb{P}_{11} + \mathbb{P}_{22} + \mathbb{P}_{33}$ is called the trace of the matrix $\mathbb{P}$) and its deviatoric (traceless) part $\mathbb{P}_s^\circ := \mathbb{P}_s - p_s\mathbb{I}$. Since $\mathbb{P}_s = p_s\mathbb{I} + \mathbb{P}_s^\circ$, $\nabla\cdot\mathbb{P}_s = \nabla p_s + \nabla\cdot\mathbb{P}_s^\circ$. So more conventionally, system (7) would be written:

$$\begin{aligned}
&\partial_t \rho_s + \nabla\cdot(\mathbf{u}_s\rho_s) = 0\\
&\partial_t(\rho_s\mathbf{u}_s) + \nabla\cdot(\rho_s\mathbf{u}_s\mathbf{u}_s) + \nabla p_s + \nabla\cdot\mathbb{P}_s^\circ = \sigma_s\mathbf{E} + \mathbf{J}_s\times\mathbf{B} + \mathbf{R}_s\\
&\partial_t\mathcal{E}_s + \nabla\cdot((\mathcal{E}_s + p_s)\mathbf{u}_s + \mathbb{P}_s^\circ\cdot\mathbf{u}_s + \mathbf{q}_s) = \mathbf{J}_s\cdot\mathbf{E} + \mathbf{R}_s\cdot\mathbf{u}_s + Q_s
\end{aligned} \tag{8}$$

The system (8) agrees exactly with the kinetic equation. The only problem is that it is not closed: the red terms are unkown unless we make an assumption about the particle distribution. Fluid closures assume that intraspecies collisions are fast enough to keep the distribution close to Maxwellian. If the distribution is Maxwellian then the red quantities, deviatoric pressure $\mathbb{P}_s^\circ$ and heat flux $\mathbf{q}_s$, will be zero. The blue terms require an interspecies collision assumption. We assume that the drag force is proportional to the interspecies drift velocity:

$$-\mathbf{R}_i = \mathbf{R}_e = e^2 n_e n_i \boldsymbol{\eta}\cdot(\mathbf{u}_i - \mathbf{u}_e), \tag{9}$$

where $\boldsymbol{\eta}$ is a proportionality constant called the **resistivity** and we have used that $\mathbf{R}_i + \mathbf{R}_e = 0$.

Since $0 = \mathbf{R}_i\cdot\mathbf{u}_i + Q_i + \mathbf{R}_e\cdot\mathbf{u}_e + Q_e$, the total heating $Q := Q_i + Q_e$ (caused by resistive drag) is $Q = -\sum\mathbf{R}_s\cdot\mathbf{u}_s \approx \mathbf{J}\cdot\boldsymbol{\eta}\cdot\mathbf{J}$, and for simplicity we can assume that resistive heating is allocated among the species in inverse proportion to the mass of each species.

Magnetohydrodynamics (MHD) regards the plasma as a single fluid and evolves *total* mass, momentum, and energy densities. The bulk fluid quantities of MHD are thus defined by summing over all species:

$$\begin{aligned} \rho &:= \rho_i + \rho_e, \\ \rho\mathbf{u} &:= \rho_i\mathbf{u}_i + \rho_e\mathbf{u}_e, \\ \mathcal{E} &:= \mathcal{E}_i + \mathcal{E}_e. \end{aligned}$$

To obtain a closed system, MHD models impose two fundamental simplifying assumptions:

1. **quasineutrality:** $n_i = n_e =: n$ (or more generally, $\sigma/e \to 0$).
2. **Ohm's law**: $\mathbf{E} = \mathbf{B} \times \mathbf{u} + \ldots$.

Ohm's law replaces electric field evolution and thus eliminates light waves from the system.

The divergence constraint $\nabla \cdot \mathbf{E} = \mu_0 c^2 e(n_i - n_e)$ says that quasineutrality is justified if $c \to \infty$ (classical, two-fluid MHD) or if $e \to \infty$ (one-fluid, possibly relativistic MHD).

One-fluid MHD additionally assumes that all species have approximately the same fluid velocity:

$$\boxed{\mathbf{u}_i \approx \mathbf{u}_e};$$

this assumption is enforced as $e \to \infty$ both by the strong electrical current $\mathbf{J} = e(\mathbf{u}_i n_i - \mathbf{u}_e n_e)$ and by the strong resistive drag $\mathbf{R}_e = e^2 n_e n_i \eta \cdot (\mathbf{u}_i - \mathbf{u}_e) = -\mathbf{R}_i$ that would otherwise result.

With this simplifying assumption, summing the system (8) over all species gives:

$$\boxed{\begin{aligned} &\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0 \\ &\partial_t(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) + \nabla p + \nabla \cdot \mathbb{P}^\circ = \sigma\mathbf{E} + \mathbf{J} \times \mathbf{B} \\ &\partial_t \mathcal{E} + \nabla \cdot ((\mathcal{E} + p)\mathbf{u} + \mathbb{P}^\circ \cdot \mathbf{u} + \mathbf{q}) = \mathbf{J} \cdot \mathbf{E} \end{aligned}} \tag{10}$$

Recall from page 18 the momentum evolution equation (5). For electrons it says:

$$\bar{\delta}_t(\rho_e \mathbf{u}_e) + \nabla \cdot \mathbb{P}_e = \sigma_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \mathbf{R}_e.$$

In the limit $e \to \infty$, the electron charge density $\sigma_e = -en_e$ becomes infinite. Assuming that the left side remains finite, dividing by σ_e makes the left side zero. Solving for **E**,

$$\mathbf{E} = \mathbf{B} \times \mathbf{u}_e + \frac{\mathbf{R}_e}{\sigma_e}.$$

In the MHD limit $n_i \approx n_e =: n$, so the current is $\mathbf{J} = en(\mathbf{u}_i - \mathbf{u}_e)$ and the drag closure (9) becomes $\mathbf{R}_e = en\eta \cdot \mathbf{J}$, i.e., $\frac{\mathbf{R}_e}{\sigma_e} = -\eta \cdot \mathbf{J}$. So Ohm's law says:

$$\boxed{\begin{aligned} \mathbf{E} =& \mathbf{B} \times \mathbf{u} && \text{(ideal term)} \\ &+ \eta \cdot \mathbf{J} && \text{(resistive term)}. \end{aligned}}$$

Classical MHD

In the classical limit, $c \to \infty$. This yields two important simplifications:

1. **Charge neutrality:**

$$\boxed{\sigma = 0.}$$

Indeed, the divergence constraint $\mu_0 \sigma = c^{-2} \nabla \cdot \mathbf{E}$ implies that $\sigma \approx 0$.

2. **Ampere's law:**

$$\boxed{\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}.}$$

Indeed, the displacement current $\partial_t \mathbf{E}$ disappears in Maxwell-Ampere:

$$\mu_0 \mathbf{J} := \nabla \times \mathbf{B} - c^{-2} \partial_t \mathbf{E}.$$

Putting it all together, we have. . .

MHD system:

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad \text{(mass continuity)},$$
$$\rho \mathrm{d}_t\mathbf{u} + \nabla p + \nabla\cdot\mathbb{P}^\circ = \mathbf{J}\times\mathbf{B} \qquad \text{(momentum balance)},$$
$$\bar{\delta}_t\mathcal{E} + \nabla\cdot(\mathbf{u}p + \mathbf{u}\cdot\mathbb{P}^\circ + \mathbf{q}) = \mathbf{J}\cdot\mathbf{E} \qquad \text{(energy balance)},$$
$$\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0 \qquad \text{(magnetic field evolution)}.$$

The divergence constraint $\nabla\cdot\mathbf{B} = 0$ is maintained by exact solutions and must be maintained in numerical solutions.

Electromagnetic closing relations:

$$\mathbf{J} := \mu_0^{-1}\nabla\times\mathbf{B} \qquad \text{(Ampere's law for current)}$$
$$\boxed{\mathbf{E} \approx \mathbf{B}\times\mathbf{u} + \boldsymbol{\eta}\cdot\mathbf{J}} \qquad \text{(Ohm's law for electric field)}$$

In a reference frame moving with the fluid, **B** remains unchanged but the electric field becomes $\mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B} = \boldsymbol{\eta}\cdot\mathbf{J}$. So Ohm's law says that, in the reference frame of the fluid, the electric field is proportional to current (i.e. to the drift velocity of the electrons). In other words, the electric field balances the resistive drag force.

Fluid closure:

$$p = \tfrac{2}{3}(\mathcal{E} - \tfrac{1}{2}\rho|\mathbf{u}|^2),$$
$$\mathbb{P}^\circ \approx -2\boldsymbol{\mu} : ((\nabla\mathbf{u})^\circ),$$
$$\mathbf{q} \approx -\mathbf{k}\cdot\nabla T;$$

$(\nabla\mathbf{u})^\circ := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - \frac{1}{3}\nabla\cdot\mathbf{u}$ is the *deviatoric strain rate*.

Closure tensors: We will neglect the viscosity $\boldsymbol{\mu}$ and heat conductivity $\mathbf{k}$. In the presence of a strong magnetic field, $\boldsymbol{\mu}$ and $\mathbf{k}$ are tensors, not scalars. In a tokamak, heat conductivity perpendicular to the magnetic field can be a million times weaker than parallel to the magnetic field, helping to confine heat. The reason is that particles spiral tightly around magnetic field lines and so easily drift along field lines. On the other hand, even when the magnetic field is strong, it is safe to assume that the resistivity $\boldsymbol{\eta}$ is a scalar (i.e., $\boldsymbol{\eta} = \eta\mathbb{I}$) and we will make this simplification.

Thermal energy evolution in MHD

To obtain a thermal energy evolution equation for MHD, we imitate the procedure for gas dynamics by subtracting kinetic energy evolution from total gas dynamic energy evolution.

Recall momentum balance:

$$\rho d_t \mathbf{u} + \nabla p + \nabla \cdot \mathbb{P}^\circ = \mathbf{J} \times \mathbf{B}.$$

Kinetic energy balance is $\mathbf{u}$ dot momentum balance:

$$\tfrac{1}{2}\rho d_t |\mathbf{u}|^2 + \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\nabla \cdot \mathbb{P}^\circ) = \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}).$$

Recall total gas-dyanamic energy balance:

$$\bar{\delta}_t \mathcal{E} + \nabla \cdot (\mathbf{u}p + \mathbf{u} \cdot \mathbb{P}^\circ + \mathbf{q}) = \mathbf{J} \cdot \mathbf{E}.$$

Subtracting kinetic energy balance from this yields thermal energy balance:

$$\boxed{\bar{\delta}_t(\tfrac{3}{2}p) + p\nabla \cdot \mathbf{u} + \mathbb{P}^\circ : \nabla \mathbf{u} + \nabla \cdot \mathbf{q} = \mathbf{J} \cdot \mathbf{E}'},$$

where we have used that thermal energy is $\frac{3}{2}$ the pressure, i.e., $\mathcal{E} = \frac{3}{2}p + \frac{1}{2}\rho|\mathbf{u}|^2$, and where $\mathbf{E}' := \mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the electric field in the reference frame of the fluid.

For resistive MHD,

$$\mathbf{E}' = \boldsymbol{\eta} \cdot \mathbf{J}.$$

Recall that

$$\bar{\delta}_t p = \partial_t p + \nabla \cdot (\mathbf{u}p) = \mathrm{d}_t p + p\nabla \cdot \mathbf{u},$$

so

$$\tfrac{3}{2}\bar{\delta}_t p + p\nabla \cdot \mathbf{u} = \tfrac{3}{2}\mathrm{d}_t p + \tfrac{5}{2}p\nabla \cdot \mathbf{u}.$$

Assuming that $\mathbb{P}^\circ = 0$ and $\mathbf{q} = 0$, pressure evolution becomes

$$\boxed{\mathrm{d}_t p + \gamma \nabla \cdot \mathbf{u} = \tfrac{2}{3}\boldsymbol{\eta} \cdot \mathbf{J}.}$$

where $\gamma := \frac{5}{3}$ is the adiabatic index.

Conservation form of MHD

A fundamental principle of physics is that total momentum and energy are conserved. This means that we should be able to put e.g. the momentum evolution equation in conservation form $\partial_t Q + \nabla \cdot \mathbf{F} = 0$.

To put **momentum evolution** in conservation form, we write the source term as a divergence using Ampere's law, vector calculus, and $\nabla \cdot \mathbf{B} = 0$:

$$
\begin{aligned}
&-\mu_0 \mathbf{J} \times \mathbf{B} = \mu_0 \mathbf{B} \times \mathbf{J} \\
&= \mathbf{B} \times \nabla \times \mathbf{B} \\
&= (\nabla \mathbf{B}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{B} \\
&= \nabla(\mathbf{B}^2/2) - \nabla \cdot (\mathbf{B}\mathbf{B}) \\
&= \nabla \cdot (\mathbb{I}\mathbf{B}^2/2 - \mathbf{B}\mathbf{B}).
\end{aligned}
$$

To put **energy evolution** in conservation form, we write the source term as a time-derivative plus a divergence, using Ampere's law, the identity $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}$, and Faraday's law:

$$
\begin{aligned}
&-\mu_0 \mathbf{E} \cdot \mathbf{J} \\
&= -\mathbf{E} \cdot \nabla \times \mathbf{B} \\
&= \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mathbf{B} \cdot \nabla \times \mathbf{E} \\
&= \nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{B} \cdot \partial_t \mathbf{B} \\
&= \nabla \cdot (\mathbf{E} \times \mathbf{B}) + \partial_t(\mathbf{B}^2/2).
\end{aligned}
$$

So **MHD in conservation form** reads

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{(mass continuity)},$$

$$\rho \mathrm{d}_t \mathbf{u} + \nabla \cdot \left(\mathbb{I}\left(p + \frac{B^2}{2\mu_0}\right) + \mu_0^{-1} \mathbf{B}\mathbf{B} + \mathbb{P}^\circ \right) = 0, \qquad \text{(momentum conservation)},$$

$$\partial_t \left(\mathcal{E} + \frac{\mathbf{B}^2}{2\mu_0} \right) + \nabla \cdot \left(\mathbf{u}(\mathcal{E} + p) + \mathbf{u} \cdot \mathbb{P}^\circ + \mathbf{q} + \mu_0^{-1} \mathbf{E} \times \mathbf{B} \right) = 0, \qquad \text{(energy conservation)},$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \qquad \text{(magnetic field evolution)},$$

where we now recognize $p_B := \frac{\mathbf{B}^2}{2\mu_0}$ as both the pressure and the energy of the magnetic field.

Notes on tensors

An n**th order tensor** has n subscripts each of which runs from 1 to 3. For example, $\mathbb{P}_{ij} = \rho\langle \mathbf{c}_i \mathbf{c}_j \rangle$ is a second-order tensor (i.e. a 3×3 matrix).

The **tensor project** of an nth order tensor A and an mth order tensor B is an $(n+m)$th order tensor $AB = A \otimes B$, where $(AB)_{i_1 \ldots i_n j_1 \ldots j_m} = A_{i_1 \ldots i_n} B_{j_1 \ldots j_m}$. For example,

$$(\mathbf{u}\mathbb{P})_{ijk} := \mathbf{u}_i \mathbb{P}_{jk}.$$

The unique second-order tensor that is invariant under rotation of coordinates is the **Kronecker delta**:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The unique third-order tensor that remain unchanged under rotation of coordinates is the Levi-Civita symbol:

$$\begin{aligned} 1 &= \epsilon_{123} = \epsilon_{231} = \epsilon_{312}, \\ -1 &= \epsilon_{213} = \epsilon_{321} = \epsilon_{132}, \text{ and} \\ 0 &= \epsilon_{ijk} \text{ if } i = j \text{ or } j = k \text{ or } i = k. \end{aligned}$$

The **Einstein summation convention** says that there is an implied sum over a repeated index in a term. A non-summed index is called a **free index**. For example, the **cross product** is defined by $(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk} \mathbf{u}_j \mathbf{v}_k$, where i is the free index.

The **dot product** of two tensors is the tensor product **contracted** (summed) over adjacent indices. E.g. $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}_i \mathbf{v}_i$ and $(\mathbf{u} \cdot \mathbb{P})_i = \mathbf{u}_j \mathbb{P}_{ji}$.

The **trace** of a tensor is its contraction over its first two indices: $\mathrm{tr}\,\mathbb{P} = \mathbb{P}_{ii}$ and $\mathbf{u} \cdot \mathbf{v} = \mathrm{tr}(\mathbf{uv})$.

The **transpose** of a matrix is defined by $M^T_{ij} = M_{ji}$.

A **symmetric matrix** M (such as the pressure tensor $\mathbb{P}$) satisfies $M^T = M$.

References

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