Evolution equations of the 13-moment model

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Derivation of the 13-moment evolution equations (without closure).

The kinetic equation

- **The kinetic equation:**

$$\boxed{\partial_t f + \nabla_{\mathbf{x}}\cdot(\mathbf{v}f) + \nabla_{\mathbf{v}}\cdot(\mathbf{a}f) = \mathcal{C}}$$

- **Lorentz force law**

$$\boxed{\mathbf{a} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)}$$

- **Collision operator**

  Conservation:

$$\int_{\mathbf{v}} \begin{bmatrix} 1 \\ \mathbf{v} \\ \|\mathbf{v}\|^2 \end{bmatrix} \mathcal{C} = 0,$$

  Entropy:

$$\int_{\mathbf{v}} f \log f\, \mathcal{C} \leq 0.$$

- **Gaussian-BGK collision operator**

  For $\mathcal{C}$ we can obtain relaxation closures with a Gaussian-BGK collision operator which relaxes toward a Gaussian distribution:

$$\mathcal{C} = \mathcal{C}_{\widetilde{\Theta}} = \frac{f_{\widetilde{\Theta}} - f}{\widetilde{\tau}},$$

  where the Gaussian distribution $f_{\widetilde{\Theta}}$ shares physically conserved moments with f and has pseudo-temperature $\widetilde{\Theta}$ equal to an affine (not necessarily convex!) combination of the pseudo-temperature Θ and its isotropization:

$$f_{\widetilde{\Theta}} = \frac{\rho \exp\left(-\mathbf{c}\cdot\widetilde{\Theta}^{-1}\cdot\mathbf{c}/2\right)}{\sqrt{\det(2\pi\widetilde{\Theta})}},$$

$$\Theta := \langle\mathbf{cc}\rangle = \int \mathbf{cc}f d\mathbf{v} \Big/ \int f d\mathbf{v},$$

$$\widetilde{\Theta} := \bar{\nu}\theta\mathbb{I} + \nu\Theta, \quad (\bar{\nu} + \nu = 1),$$

$$\bar{\nu} := 1/\,\mathrm{Pr} = \tau/\widetilde{\tau}.$$

Here $\widetilde{\tau}$ is the heat flux relaxation period, τ is the relaxation period of deviatoric pressure, and $C_{\widetilde{\Theta}}$ respects entropy if $\widetilde{\Theta}$ is positive definite (i.e. $0 < \bar{\nu} \leq 3/2$). In the limit $\bar{\nu} \searrow 0$ heat flux goes to zero and the solution approximates hyperbolic Gaussian-moment (10-moment) gas dynamics.

Use of a Gaussian-BKG collision operator allows one to tune the viscosity $\mu = p\tau$ and the thermal conductivity $k = \frac{5}{2}\frac{\mu}{m\,\mathrm{Pr}}$.

[Products and powers are uncontracted tensor operations.]

Conserved moments are moments of monomials in $\mathbf{v}$. Let $\chi = \chi(\mathbf{v})$. Take the χth moment of the kinetic equation:

$$\int_{\mathbf{v}} \chi\left(\partial_t f + \nabla_{\mathbf{x}}\cdot(\mathbf{v}f) + \nabla_{\mathbf{v}}\cdot(\mathbf{a}f) = \mathcal{C}\right)$$

Integrate by parts to get

$$\begin{aligned} &\partial_t \int_{\mathbf{v}} \chi f + \nabla\cdot\int_{\mathbf{v}} \mathbf{v}\chi f \\ &\quad = \int_{\mathbf{v}} f\mathbf{a}\cdot\nabla_{\mathbf{v}}\chi + \int_{\mathbf{v}} \chi\mathcal{C} \end{aligned} \qquad (1)$$

Choose $\chi = \mathbf{v}^n$. Define

$$\widetilde{\mathbf{F}}^n := \int_{\mathbf{v}} \mathbf{v}^n f,$$

$$\widetilde{\mathbf{C}}^n := \int_{\mathbf{v}} \mathbf{v}^n \mathcal{C}.$$

Then

$$\mathbf{a}\cdot\nabla_{\mathbf{v}}\chi = \frac{q}{m} n \,\mathrm{Sym}\left(\mathbf{E}\mathbf{v}^{n-1} + \mathbf{v}^n \times \mathbf{B}\right).$$

Substituting into equation (1) gives:

General conserved moment evolution

$$\boxed{\partial_t \widetilde{\mathbf{F}}^n + \nabla\cdot\widetilde{\mathbf{F}}^{n+1} = \frac{q}{m} n \,\mathrm{Sym}\left(\mathbf{E}\widetilde{\mathbf{F}}^{n-1} + \widetilde{\mathbf{F}}^n \times \mathbf{B}\right) + \widetilde{\mathbf{C}}^n}$$

This is a hierarchy of conserved moment evolution equations:

$$\begin{aligned} \partial_t \widetilde{\mathbf{F}}^0 + \nabla\cdot\widetilde{\mathbf{F}}^1 &= 0, \\ \partial_t \widetilde{\mathbf{F}}^1 + \nabla\cdot\widetilde{\mathbf{F}}^2 &= \frac{q}{m}(\mathbf{E}\widetilde{\mathbf{F}}^0 + \widetilde{\mathbf{F}}^1 \times \mathbf{B}), \\ \partial_t \widetilde{\mathbf{F}}^2 + \nabla\cdot\widetilde{\mathbf{F}}^3 &= \frac{q}{m} 2\,\mathrm{Sym}(\mathbf{E}\widetilde{\mathbf{F}}^1 + \widetilde{\mathbf{F}}^2 \times \mathbf{B}) + \widetilde{\mathbf{C}}^2 \\ \partial_t \widetilde{\mathbf{F}}^3 + \nabla\cdot\widetilde{\mathbf{F}}^4 &= \frac{q}{m} 3\,\mathrm{Sym}(\mathbf{E}\widetilde{\mathbf{F}}^2 + \widetilde{\mathbf{F}}^3 \times \mathbf{B}) + \widetilde{\mathbf{C}}^3. \end{aligned}$$

Denote traces of conserved moments by

$$\widetilde{\mathbf{F}}^n_{\mathrm{tr}} = \mathrm{tr}\,\widetilde{\mathbf{F}}^n, \quad \widetilde{\mathbf{C}}^n_{\mathrm{tr}} = \mathrm{tr}\,\widetilde{\mathbf{C}}^n$$

Taking the trace of the equations for $\widetilde{\mathbf{F}}^2$ and $\widetilde{\mathbf{F}}^3$ yields

$$\begin{aligned} \partial_t \widetilde{\mathbf{F}}^2_{\mathrm{tr}} + \nabla\cdot\widetilde{\mathbf{F}}^3_{\mathrm{tr}} &= \frac{q}{m} 2\mathbf{E}\cdot\widetilde{\mathbf{F}}^3_{\mathrm{tr}} + \widetilde{\mathbf{C}}^2_{\mathrm{tr}}, \\ \partial_t \widetilde{\mathbf{F}}^3_{\mathrm{tr}} + \nabla\cdot\widetilde{\mathbf{F}}^4_{\mathrm{tr}} &= \frac{q}{m}\left[2\mathbf{E}\cdot\widetilde{\mathbf{F}}^2 + \mathrm{Sym}(\mathbf{E}\widetilde{\mathbf{F}}^2_{\mathrm{tr}}) + \mathrm{Sym}(\widetilde{\mathbf{F}}^3_{\mathrm{tr}} \times \mathbf{B})\right] + \widetilde{\mathbf{C}}^3_{\mathrm{tr}}, \end{aligned}$$

where we have used that

$$\begin{aligned} \mathrm{tr}\, n\,\mathrm{Sym}(\mathbf{E}\widetilde{\mathbf{F}}^{n-1}) &= 2\mathbf{E}\cdot\widetilde{\mathbf{F}}^{n-1} + (n-2)\,\mathrm{Sym}(\mathbf{E}\,\mathrm{tr}\,\widetilde{\mathbf{F}}^{n-1}), \\ \mathrm{tr}\, n\,\mathrm{Sym}(\widetilde{\mathbf{F}}^n \times \mathbf{B}) &= (n-2)\,\mathrm{Sym}(\mathrm{tr}\,\widetilde{\mathbf{F}}^n \times \mathbf{B}). \end{aligned}$$

To summarize:

13-moment evolution equations in conservation form

$$
\begin{aligned}
&\partial_t\widetilde{\mathbf{F}}^0 + \nabla\cdot\widetilde{\mathbf{F}}^1 = \widetilde{\mathbf{C}}^0 = 0,\\
&\partial_t\widetilde{\mathbf{F}}^1 + \nabla\cdot\widetilde{\mathbf{F}}^2 = \tfrac{q}{m}(\mathbf{E}\widetilde{\mathbf{F}}^0 + \widetilde{\mathbf{F}}^1 \times \mathbf{B}) + \widetilde{\mathbf{C}}^1,\\
&\partial_t\widetilde{\mathbf{F}}^2 + \nabla\cdot\widetilde{\mathbf{F}}^3 = \tfrac{q}{m}2\,\mathrm{Sym}(\mathbf{E}\widetilde{\mathbf{F}}^1 + \widetilde{\mathbf{F}}^2 \times \mathbf{B}) + \widetilde{\mathbf{C}}^2\\
&\partial_t\widetilde{\mathbf{F}}^3_{\mathrm{tr}} + \nabla\cdot\widetilde{\mathbf{F}}^4_{\mathrm{tr}} = \tfrac{q}{m}\left[2\mathbf{E}\cdot\widetilde{\mathbf{F}}^2 + \mathrm{Sym}(\mathbf{E}\widetilde{\mathbf{F}}^2_{\mathrm{tr}}) + \mathrm{Sym}(\widetilde{\mathbf{F}}^3_{\mathrm{tr}} \times \mathbf{B})\right] + \widetilde{\mathbf{C}}^3_{\mathrm{tr}}
\end{aligned}
\tag{2}
$$

where

$$
\widetilde{\mathbf{F}}^n = \int_{\mathbf{v}} \mathbf{v}^n f, \quad \widetilde{\mathbf{F}}^n_{\mathrm{tr}} = \int_{\mathbf{v}} \mathrm{tr}\,\mathbf{v}^n f, \quad \widetilde{\mathbf{C}}^n = \int_{\mathbf{v}} \mathbf{v}^n \mathcal{C}.
$$

The challenge is to find closures for $\widetilde{\mathbf{F}}^3$ and $\widetilde{\mathbf{F}}^4_{\mathrm{tr}}$ in terms of the evolved moments for which the operator on the left hand side maintains hyperbolicity.

Closures for the collisional terms come from the assumed collision operator. Anticipating, for Gaussian-BGK,

$$
\widetilde{\mathbf{C}}^1 = 0, \qquad \widetilde{\mathbf{C}}^2 = \frac{-(\int_{\mathbf{c}} \mathbf{c}\mathbf{c} f)^\circ}{\tau} \quad \text{where } \tau := \mathrm{Pr}\,\widetilde{\tau}, \qquad \widetilde{\mathbf{C}}^3_{\mathrm{tr}} = \frac{-\int_{\mathbf{c}} |\mathbf{c}|^2 \mathbf{c} f}{\widetilde{\tau}}
$$

Primitive moments

- **Fluid closures should be Galilean invariant.**
  In the classical approximation, under a Galilean shift $\mathbf{B}' = \mathbf{B}$ and $\mathbf{E}' = \mathbf{E} + (\mathbf{v} - \mathbf{v}') \times \mathbf{B}$ so that the Lorentz force is invariant. The kinetic equation is therefore Galilean-invariant, so we require fluid closures to be Galilean-invariant.
- **Primitive moments are Galilean-invariant**

  Definitions:

$$\rho = \int_v f$$

$$\langle \chi \rangle := \frac{\int_\mathbf{v} \chi}{\rho}$$

$$\mathbf{u} := \langle \mathbf{v} \rangle$$

$$\mathbf{c} := \mathbf{v} - \mathbf{u}$$

$$\mathbf{F}^n := \int_\mathbf{v} \mathbf{c}^n f$$

  Specifying closure in terms of primitive moments $\mathbf{F}^n$ ensures that closures are Galilean-invariant.

Eliminating primitive variables

To express primitive moments in terms of conserved moments we observe that

$$c^n = \mathrm{Sym}[c^n] = \mathrm{Sym}[(v-u)^n]$$
$$= \mathrm{Sym}\sum_{j=0}^{n}(-1)^j\binom{n}{j}u^j v^{n-j}.$$

Primitive moments are thus given in terms of conserved moments by

$$\mathbf{F}^n = \mathrm{Sym}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\mathbf{u}^j\widetilde{\mathbf{F}}^{n-j}$$
$$= \widetilde{\mathbf{F}}^n + \mathrm{Sym}\sum_{j=1}^{n-2}(-1)^j\binom{n}{j}\mathbf{u}^j\widetilde{\mathbf{F}}^{n-j}$$
$$+ (-1)^n(1-n)\rho\mathbf{u}^n.$$

That is,

$$\mathbf{F}^0 = \rho,$$
$$\mathbf{F}^1 = \mathbf{0},$$
$$\mathbf{F}^2 = \widetilde{\mathbf{F}}^2 - \rho\mathbf{u}^2,$$
$$\mathbf{F}^3 = \widetilde{\mathbf{F}}^3 - \mathrm{Sym}(3\mathbf{u}\widetilde{\mathbf{F}}^2) + 2\rho\mathbf{u}^3,$$
$$\mathbf{F}^4 = \widetilde{\mathbf{F}}^4 - \mathrm{Sym}\left(4\mathbf{u}\widetilde{\mathbf{F}}^3 - 6\mathbf{u}^2\widetilde{\mathbf{F}}^2\right) - 3\rho\mathbf{u}^4$$
$$\mathbf{F}^5 = \widetilde{\mathbf{F}}^5 - \mathrm{Sym}\left(5\mathbf{u}\widetilde{\mathbf{F}}^4 - 10\mathbf{u}^2\widetilde{\mathbf{F}}^3 + 10\mathbf{u}^3\widetilde{\mathbf{F}}^2\right) + 4\rho\mathbf{u}^5$$

Taking traces,

$$\mathbf{F}^2_{\mathrm{tr}} = \widetilde{\mathbf{F}}^2_{\mathrm{tr}} - \rho|\mathbf{u}|^2,$$
$$\mathbf{F}^3_{\mathrm{tr}} = \widetilde{\mathbf{F}}^3_{\mathrm{tr}} - \widetilde{\mathbf{F}}^2_{\mathrm{tr}}\mathbf{u} - 2\mathbf{u}\cdot\widetilde{\mathbf{F}}^2 + 2\rho|\mathbf{u}|^2\mathbf{u},$$
$$\mathbf{F}^4_{\mathrm{tr}} = \widetilde{\mathbf{F}}^4_{\mathrm{tr}} - 2\,\mathrm{Sym}(\mathbf{u}\widetilde{\mathbf{F}}^3_{\mathrm{tr}}) - 2\mathbf{u}\cdot\widetilde{\mathbf{F}}^3$$
$$+ \mathbf{u}^2\widetilde{\mathbf{F}}^2_{\mathrm{tr}} + |\mathbf{u}|^2\widetilde{\mathbf{F}}^2 + 4\,\mathrm{Sym}(\mathbf{u}^2\cdot\widetilde{\mathbf{F}}^2) - 3\rho|\mathbf{u}|^2\mathbf{u}^2.$$

Solve for the higher conserved moments $\widetilde{\mathbf{F}}^3$ and $\widetilde{\mathbf{F}}^4_{\mathrm{tr}}$ and substitute into the 13-moment system (2). Assuming $\mathbf{E} = 0 = \mathbf{B}$ and $\widetilde{\mathbf{C}}^2 = 0 = \widetilde{\mathbf{C}}^3_{\mathrm{tr}}$, we get:

13-moment system in evolved conserved moments

$$\partial_t\widetilde{\mathbf{F}}^0 + \nabla\cdot\widetilde{\mathbf{F}}^1 = 0,$$
$$\partial_t\widetilde{\mathbf{F}}^1 + \nabla\cdot\widetilde{\mathbf{F}}^2 = 0,$$
$$\partial_t\widetilde{\mathbf{F}}^2 + \nabla\cdot\left(\mathrm{Sym}\left(3\mathbf{u}\widetilde{\mathbf{F}}^2\right) - 2\rho\mathbf{u}^3\right) + \nabla\cdot\mathbf{F}^3 = 0,$$
$$\partial_t\widetilde{\mathbf{F}}^3_{\mathrm{tr}} + \nabla\cdot\left(2\,\mathrm{Sym}\left(\mathbf{u}\widetilde{\mathbf{F}}^3_{\mathrm{tr}}\right) + 2\mathbf{u}\cdot\widetilde{\mathbf{F}}^3\right)$$
$$- \nabla\cdot\left(\mathbf{u}^2\widetilde{\mathbf{F}}^2_{\mathrm{tr}} + |\mathbf{u}|^2\widetilde{\mathbf{F}}^2 + 4\,\mathrm{Sym}\left(\mathbf{u}^2\cdot\widetilde{\mathbf{F}}^2\right)\right)$$
$$+ \nabla\cdot\left(3\rho|\mathbf{u}|^2\mathbf{u}^2\right) + \nabla\cdot\mathbf{F}^4_{\mathrm{tr}} = 0.$$

We seek closures for the primitive moments $\mathbf{F}^3$ and $\mathbf{F}^4_{\mathrm{tr}}$ in terms of the lower primitive moments ρ, $\mathbf{F}^2$, and $\mathbf{F}^3_{\mathrm{tr}}$.

Eliminating conserved variables

To express conserved moments in terms of primitive moments we observe that

$$v^n = \mathrm{Sym}[v^n] = \mathrm{Sym}[(u+c)^n]$$
$$= \mathrm{Sym}\sum_{j=0}^{n}\binom{n}{j} u^j c^{n-j}.$$

Conserved moments are thus given in terms of primitive moments by

$$\widetilde{\mathbf{F}}^n = \mathrm{Sym}\sum_{j=0}^{n}\binom{n}{j}\mathbf{u}^j\mathbf{F}^{n-j}$$
$$=\mathbf{F}^n + \mathrm{Sym}\sum_{j=1}^{n-2}\binom{n}{j}\mathbf{u}^j\mathbf{F}^{n-j} + \rho\mathbf{u}^n,$$

That is,

$$\widetilde{\mathbf{F}}^0 = \rho,$$
$$\widetilde{\mathbf{F}}^1 = \rho\mathbf{u},$$
$$\widetilde{\mathbf{F}}^2 = \rho\mathbf{u}^2 + \mathbf{F}^2,$$
$$\widetilde{\mathbf{F}}^3 = \rho\mathbf{u}^3 + \mathrm{Sym}\left(3\mathbf{u}\mathbf{F}^2\right) + \mathbf{F}^3,$$
$$\widetilde{\mathbf{F}}^4 = \rho\mathbf{u}^4 + \mathrm{Sym}\left(6\mathbf{u}^2\mathbf{F}^2 + 4\mathbf{u}\mathbf{F}^3\right) + \mathbf{F}^4$$
$$\widetilde{\mathbf{F}}^5 = \rho\mathbf{u}^5 + \mathrm{Sym}\left(10\mathbf{u}^3\mathbf{F}^2 + 10\mathbf{u}^2\mathbf{F}^3 + 5\mathbf{u}\mathbf{F}^4\right) + \mathbf{F}^5,$$

Taking traces,

$$\widetilde{\mathbf{F}}^2_{\mathrm{tr}} = \rho|\mathbf{u}|^2 + \mathbf{F}^2_{\mathrm{tr}},$$
$$\widetilde{\mathbf{F}}^3_{\mathrm{tr}} = \rho|\mathbf{u}|^2\mathbf{u} + \mathbf{F}^2_{\mathrm{tr}}\mathbf{u} + 2\mathbf{u}\cdot\mathbf{F}^2 + \mathbf{F}^3_{\mathrm{tr}},$$
$$\widetilde{\mathbf{F}}^4_{\mathrm{tr}} = \rho|\mathbf{u}|^2\mathbf{u}^2 + \mathbf{u}^2\mathbf{F}^2_{\mathrm{tr}} + |\mathbf{u}|^2\mathbf{F}^2 + 4\,\mathrm{Sym}(\mathbf{u}^2\cdot\mathbf{F}^2)$$
$$+ 2\,\mathrm{Sym}(\mathbf{u}\mathbf{F}^3_{\mathrm{tr}}) + 2\mathbf{u}\cdot\mathbf{F}^3 + \mathbf{F}^4_{\mathrm{tr}}$$

Use these relations to eliminate conserved variables. Get:

13-moment evolution of primitive variables in conservation form

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0,$$
$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\left(\rho\mathbf{u}^2 + \mathbf{F}^2\right) = 0,$$
$$\partial_t\left(\rho\mathbf{u}^2 + \mathbf{F}^2\right) + \nabla\cdot\left(\rho\mathbf{u}^3 + \mathrm{Sym}\left(3\mathbf{u}\mathbf{F}^2\right) + \mathbf{F}^3\right) = 0,$$
$$\partial_t\left(\rho|\mathbf{u}|^2\mathbf{u} + \mathbf{F}^2_{\mathrm{tr}}\mathbf{u} + 2\mathbf{u}\cdot\mathbf{F}^2 + \mathbf{F}^3_{\mathrm{tr}}\right) + \nabla\cdot\left(\rho|\mathbf{u}|^2\mathbf{u}^2\right)$$
$$+ \nabla\cdot\left(\mathbf{u}^2\mathbf{F}^2_{\mathrm{tr}} + |\mathbf{u}|^2\mathbf{F}^2 + 4\,\mathrm{Sym}(\mathbf{u}^2\cdot\mathbf{F}^2)\right)$$
$$+ \nabla\cdot\left(2\,\mathrm{Sym}(\mathbf{u}\mathbf{F}^3_{\mathrm{tr}}) + 2\mathbf{u}\cdot\mathbf{F}^3\right) + \nabla\cdot\mathbf{F}^4_{\mathrm{tr}} = 0. \quad (3)$$

Mapping onto Torrilhon

To map onto [Torrilhon09], make the definitions

$$\underline{\underline{p}} = \mathbf{F}^2, \quad 2\mathbf{q} = \mathbf{F}^3_{\mathrm{tr}},$$
$$3p = \mathbf{F}^2_{\mathrm{tr}}, \quad \underline{\underline{\underline{m}}} = \mathbf{F}^3, \quad \underline{\underline{R}} = \mathbf{F}^4_{\mathrm{tr}}.$$

Then system (3) reads

$$\begin{aligned}
&\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\partial_t(\rho\mathbf{u}) + \nabla\cdot\left(\rho\mathbf{u}^2 + \underline{\underline{p}}\right) = 0,\\
&\partial_t\left(\rho\mathbf{u}^2 + \underline{\underline{p}}\right) + \nabla\cdot\left(\rho\mathbf{u}^3 + \mathrm{Sym}\,(3\mathbf{u}\underline{\underline{p}}) + \underline{\underline{\underline{m}}}\right) = 0,\\
&\partial_t\left(\rho|\mathbf{u}|^2\mathbf{u} + 3p\mathbf{u} + 2\mathbf{u}\cdot\underline{\underline{p}} + 2\mathbf{q}\right)\\
&\qquad + \nabla\cdot\left(\rho|\mathbf{u}|^2\mathbf{u}^2 + 3\mathbf{u}^2 p + |\mathbf{u}|^2\underline{\underline{p}} + 4\,\mathrm{Sym}(\mathbf{u}^2\cdot\underline{\underline{p}}) + 4\,\mathrm{Sym}(\mathbf{u}\mathbf{q}) + 2\mathbf{u}\cdot\underline{\underline{\underline{m}}} + \underline{\underline{R}}\right) = 0,
\end{aligned}$$

which, after dividing the last equation by 2, concurs with system (4.16) in [Torrilhon09].

Primitive moment evolution equations

One can also express the evolution equations entirely in terms of primitive variables. While it is possible to do this by combining equations, probably an easier way is to take primitive moments of the kinetic equation directly.

Relations for primitive moments:

$$\mathbf{c}(t,\mathbf{x},\mathbf{v}) := \mathbf{v} - \mathbf{u}(t,\mathbf{x})$$

$$\int_{\mathbf{v}} = \int_{\mathbf{c}}$$

$$\chi(t,\mathbf{x},\mathbf{v}) = \chi(\mathbf{c}) = \mathbf{c}^n$$

$$\nabla_{\mathbf{v}}\chi = \nabla_{\mathbf{c}}\chi$$

$$\mathrm{d}_t^{\mathbf{v}} := \partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}}$$

$$\rho\langle\alpha\rangle := \int_{\mathbf{v}}\alpha$$

$$\bar{D}_t(\alpha) := \partial_t\alpha + \nabla_{\mathbf{x}}\cdot(\mathbf{v}\alpha) + \nabla_{\mathbf{v}}\cdot(\mathbf{a}\alpha)$$

$$D_t := \partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} + \mathbf{a}\cdot\nabla_{\mathbf{v}} = \bar{D}_t$$

$$\bar{D}_t(\alpha\beta) = (\bar{D}_t\alpha)\beta + \alpha D_t\beta$$

Multiply the kinetic equation

$$\bar{D}_t f = C$$

by χ to get

$$\boxed{\bar{D}_t(\chi f) = f D_t\chi + \chi\mathcal{C}.} \tag{4}$$

But observe that for $\chi(\mathbf{c})$:

$$D_t = \mathrm{d}_t^{\mathbf{v}} + \mathbf{a}\cdot\nabla_{\mathbf{v}}\cdot,$$

$$\mathrm{d}_t^{\mathbf{v}}\chi = (\mathrm{d}_t^{\mathbf{v}}\mathbf{c})\cdot\nabla_{\mathbf{c}}\chi$$

$$= -(\mathrm{d}_t^{\mathbf{v}}\mathbf{u})\cdot\nabla_{\mathbf{c}}\chi,$$

$$\mathrm{d}_t^{\mathbf{v}} = \mathrm{d}_t^{\mathbf{u}} + \mathbf{c}\cdot\nabla_{\mathbf{x}};$$

putting it together,

$$D_t\chi = \left(\mathbf{a} - \mathrm{d}_t^{\mathbf{u}}\mathbf{u} - \mathbf{c}\cdot\nabla_{\mathbf{x}}\mathbf{u}\right)\cdot\nabla_{\mathbf{c}}\chi \tag{5}$$

But solving momentum evolution

$$\rho\,\mathrm{d}_t^{\mathbf{u}}\mathbf{u} + \nabla\cdot\mathbb{F}^2 = \rho\langle\mathbf{a}\rangle + \mathbf{C}^1$$

for $\mathrm{d}_t^{\mathbf{u}}\mathbf{u}$, substituting in (5), and defining

$$\mathbf{a}' := \mathbf{a} - \langle\mathbf{a}\rangle = \frac{q}{m}\mathbf{c}\times\mathbf{B}$$

gives

$$\boxed{D_t\chi = \left(\mathbf{a}' - \mathbf{c}\cdot\nabla_{\mathbf{x}}\mathbf{u}\right)\cdot\nabla_{\mathbf{c}}\chi + \frac{\nabla\cdot\mathbb{F}^2 - \mathbf{C}^1}{\rho}\cdot\nabla_{\mathbf{c}}\chi.} \tag{6}$$

Substituting (6) into the kinetic equation (4) and integrating over velocity space yields a generic evolution equation for primitive moments:

$$\partial_t(\rho\langle\chi\rangle) + \nabla\cdot(\rho\mathbf{u}\langle\chi\rangle) + \nabla\cdot(\rho\langle\mathbf{c}\chi\rangle) = (\nabla\cdot\mathbb{F}^2 - \mathbf{C}^1)\cdot\langle\nabla_{\mathbf{c}}\chi\rangle + \rho\left\langle\left(\mathbf{a}' - \mathbf{c}\cdot\nabla\mathbf{u}\right)\cdot\nabla_{\mathbf{c}}\chi\right\rangle + \int_{\mathbf{v}}\chi\mathcal{C}. \tag{7}$$

Now impose that $\chi(\mathbf{c}) = \mathbf{c}^n$. For a generic $\underline{\alpha}$, $\underline{\alpha}\cdot\nabla_{\mathbf{c}}\left(\mathbf{c}^n\right) = n\,\mathrm{Sym}\left(\underline{\alpha}\mathbf{c}^{n-1}\right)$. So

$$\rho\left\langle(\mathbf{a}' - \mathbf{c}\cdot\nabla\mathbf{u})\cdot\nabla_{\mathbf{c}}\mathbf{c}^n\right\rangle = n\rho\,\mathrm{Sym}\left\langle(\mathbf{a}' - \mathbf{c}\cdot\nabla\mathbf{u})\mathbf{c}^{n-1}\right\rangle = n\,\mathrm{Sym}\left(\frac{q}{m}\mathbb{F}^n\times\mathbf{B} - \mathbb{F}^n\cdot\nabla\mathbf{u}\right) \tag{8}$$

Primitive moment evolution equations

Substituting identity (8) into equation (7) gives an evolution equation for primitive moments:

$$\boxed{\bar{\delta}_t \mathbf{F}^n + n\,\mathrm{Sym}\left(\overline{\mathbf{F}}^{n-1}(\mathbf{C}^1 - \nabla\cdot\mathbf{F}^2) + \mathbf{F}^n\cdot\nabla\mathbf{u}\right) + \nabla\cdot\mathbf{F}^{n+1} = n\,\mathrm{Sym}\left(\tfrac{q}{m}\mathbf{F}^n \times \mathbf{B}\right) + \mathbf{C}^n},$$

where

$$\overline{\mathbf{F}}^n := \langle \mathbf{c}^n \rangle = \mathbf{F}^n/\rho$$

is a generalized pseudo-temperature tensor. For $n = 2$ and $n = 3$ this says:

$$\bar{\delta}_t \mathbf{F}^2 + 2\,\mathrm{Sym}\left(\mathbf{F}^2\cdot\nabla\mathbf{u}\right) + \nabla\cdot\mathbf{F}^3 = 2\,\mathrm{Sym}\left(\tfrac{q}{m}\mathbf{F}^2 \times \mathbf{B}\right) + \mathbf{C}^2,$$

$$\bar{\delta}_t \mathbf{F}^3 + 3\,\mathrm{Sym}\left(\overline{\mathbf{F}}^2(\mathbf{C}^1 - \nabla\cdot\mathbf{F}^2) + \mathbf{F}^3\cdot\nabla\mathbf{u}\right) + \nabla\cdot\mathbf{F}^4 = 3\,\mathrm{Sym}\left(\tfrac{q}{m}\mathbf{F}^3 \times \mathbf{B}\right) + \mathbf{C}^3.$$

To facilitate taking the trace, use the general relations

$$\mathrm{tr}\, n\,\mathrm{Sym}(M^n\cdot A^2) = (n-2)\,\mathrm{Sym}(M^n_{\mathrm{tr}}\cdot A^2) + 2M^n : A^2$$

$$\mathrm{tr}\, n\,\mathrm{Sym}(M^{n-1}A^1) = (n-2)\,\mathrm{Sym}(M^{n-1}_{\mathrm{tr}}A^1) + 2\,\mathrm{Sym}(M^{n-1}\cdot A^1)$$

$$\mathrm{tr}\, n\,\mathrm{Sym}(M^n \times B) = (n-2)\,\mathrm{Sym}(M^n_{\mathrm{tr}} \times B)$$

Taking traces gives

$$\bar{\delta}_t \mathbf{F}^2_{\mathrm{tr}} + 2\mathbf{F}^2 : \nabla\mathbf{u} + \nabla\cdot\mathbf{F}^3_{\mathrm{tr}} = \mathbf{C}^2_{\mathrm{tr}},$$

$$\bar{\delta}_t \mathbf{F}^3_{\mathrm{tr}} + (\mathbb{I}\overline{\mathbf{F}}^2_{\mathrm{tr}} + 2\overline{\mathbf{F}}^2)\cdot(\mathbf{C}^1 - \nabla\cdot\mathbf{F}^2) + \mathbf{F}^3_{\mathrm{tr}}\cdot\nabla\mathbf{u} + 2\mathbf{F}^3 : \nabla\mathbf{u} + \nabla\cdot\mathbf{F}^4_{\mathrm{tr}} = \tfrac{q}{m}\mathbf{F}^3_{\mathrm{tr}} \times \mathbf{B} + \mathbf{C}^3_{\mathrm{tr}}.$$

To summarize:

13-moment primitive moment evolution equations

$$
\begin{aligned}
&\bar{\delta}_t \rho = \mathbf{C}^0,\\
&\rho \mathrm{d}_t \mathbf{u} + \nabla\cdot\mathbf{F}^2 = \tfrac{q}{m}\rho\,(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + \mathbf{C}^1,\\
&\bar{\delta}_t \mathbf{F}^2 + 2\,\mathrm{Sym}\,(\mathbf{F}^2\cdot\nabla\mathbf{u}) + \nabla\cdot\mathbf{F}^3 = 2\,\mathrm{Sym}\,(\tfrac{q}{m}\mathbf{F}^2\times\mathbf{B}) + \mathbf{C}^2,\\
&\bar{\delta}_t \mathbf{F}^3_{\mathrm{tr}} + (\mathbb{I}\bar{\mathbf{F}}^2_{\mathrm{tr}} + 2\bar{\mathbf{F}}^2)\cdot(\mathbf{C}^1 - \nabla\cdot\mathbf{F}^2) + \mathbf{F}^3_{\mathrm{tr}}\cdot\nabla\mathbf{u} + 2\mathbf{F}^3 : \nabla\mathbf{u} + \nabla\cdot\mathbf{F}^4_{\mathrm{tr}} = \tfrac{q}{m}\mathbf{F}^3_{\mathrm{tr}}\times\mathbf{B} + \mathbf{C}^3_{\mathrm{tr}}.
\end{aligned}
$$

To map onto [Torrilhon09], use/recall the definitions

$$
\underline{\underline{\Theta}} = \bar{\mathbf{F}}^2, \qquad \underline{\underline{p}} = \mathbf{F}^2, \qquad 2\mathbf{q} = \mathbf{F}^3_{\mathrm{tr}},
$$

$$
3\theta = \bar{\mathbf{F}}^2_{\mathrm{tr}}, \qquad 3p = \mathbf{F}^2_{\mathrm{tr}}, \qquad \underline{\underline{\underline{m}}} = \mathbf{F}^3, \qquad \underline{\underline{R}} = \mathbf{F}^4_{\mathrm{tr}}.
$$

13-moment primitive moment evolution equations (Torrilhon's notation)

$$
\begin{aligned}
&\bar{\delta}_t \rho = \mathbf{C}^0,\\
&\rho \mathrm{d}_t \mathbf{u} + \nabla\cdot\underline{\underline{p}} = \tfrac{q}{m}\rho\,(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + \mathbf{C}^1,\\
&\bar{\delta}_t \underline{\underline{p}} + 2\,\mathrm{Sym}\,(\underline{\underline{p}}\cdot\nabla\mathbf{u}) + \nabla\cdot\underline{\underline{\underline{m}}} = 2\,\mathrm{Sym}\,(\tfrac{q}{m}\underline{\underline{p}}\times\mathbf{B}) + \mathbf{C}^2,\\
&\bar{\delta}_t (2\mathbf{q}) + (3\theta\mathbb{I} + 2\underline{\underline{\Theta}})\cdot(\mathbf{R} - \nabla\cdot\underline{\underline{p}}) + 2\mathbf{q}\cdot\nabla\mathbf{u} + 2\underline{\underline{\underline{m}}} : \nabla\mathbf{u} + \nabla\cdot\underline{\underline{R}} = \tfrac{q}{m}2\mathbf{q}\times\mathbf{B} + \mathbf{C}^3_{\mathrm{tr}}.
\end{aligned}
$$