Accurate simulation of fast magnetic reconnection calls for higher-moment fluid models.

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Abstract: This talk argues that the simplest fluid model of plasma that can resolve steady fast magnetic reconnection is a hyperbolic two-fluid plasma model that evolves at least 13 moments in each species (i.e. including tensor pressure and heat flux). Models with fewer moments that admit fast reconnection involve physical deficiencies (anomalous closures) or numerical difficulties (due to complex diffusive closures or dispersive terms). A key point is that higher-moment hyperbolic models are the simplest fluid models that behave analogously to kinetic models, facilitating asymptotic-preserving kinetic-fluid stitching.

Outline

Fundamental model: particle-Maxwell (relativistic)

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0,$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0.$$

- **Charge moments:**

$$\sigma := \sum_p S_p(\mathbf{x}_p) q_p,$$
$$\mathbf{J} := \sum_p S_p(\mathbf{x}_p) q_p v_p,$$

- **Particle equations:**

$$d_t \mathbf{x}_p = \mathbf{v}_p,$$
$$d_t(\gamma_p \mathbf{v}_p) = \mathbf{a}_\mathrm{p}(\mathbf{x}_p, \mathbf{v}_p),$$
$$\gamma_p^{-2} := 1 - (\mathbf{v}_p/c)^2.$$

- **Lorentz acceleration**

$$\mathbf{a}_\mathrm{p}(\mathbf{x}, \mathbf{v}) = \frac{q_p}{m_p} \left(\mathbf{E}(\mathbf{x}) + \mathbf{v} \times \mathbf{B}(\mathbf{x}) \right)$$

Fundamental collisionless parameters

Physical constants that define an ion-electron plasma:

1. e (charge of proton),
2. m_i, m_e (ion and electron mass),
3. c (speed of light).

Fundamental parameters that characterize the state of a plasma:

1. n_0 (typical particle density),
2. T_0 (typical temperature),
3. B_0 (typical magnetic field).

Derived quantities:

- $p_0 := n_0 T_0$ (thermal pressure)
- $p_B := \frac{B_0^2}{2\mu_0}$ (magnetic pressure)
- $\rho_s := n_0 m_s$ (typical density).

Subsidiary time, velocity, and space scale parameters:

$$\textbf{plasma frequencies:} \quad \omega_{p,s}^2 := \frac{n_0 e^2}{\epsilon_0 m_s},$$

$$\textbf{gyrofrequencies:} \quad \omega_{g,s} := \frac{eB_0}{m_s},$$

$$\textbf{thermal velocities:} \quad v_{t,s}^2 := \frac{2p_0}{\rho_s},$$

$$\textbf{Alfvén speeds:} \quad v_{A,s}^2 := \frac{2p_B}{\rho_s} = \frac{B_0^2}{\mu_0 m_s n_0},$$

$$\textbf{Debye length:} \quad \lambda_D := \frac{v_{t,s}}{\omega_{p,s}} = \sqrt{\frac{\epsilon_0 T_0}{n_0 e^2}},$$

$$\textbf{gyroradii:} \quad r_{g,s} := \frac{v_{t,s}}{\omega_{g,s}} = \frac{m_s v_{t,s}}{eB_0},$$

$$\textbf{skin depths:} \quad \delta_s := \frac{v_{A,s}}{\omega_{g,s}} = \frac{c}{\omega_{p,s}} = \sqrt{\frac{m_s}{\mu_0 n_s e^2}}.$$

$$\text{plasma } \beta := \frac{p_0}{p_B} = \left(\frac{v_{t,s}}{v_{A,s}}\right)^2 = \left(\frac{r_{g,s}}{\delta_s}\right)^2.$$

$$\text{another ratio } := \frac{c}{v_{A,s}} = \frac{r_{g,s}}{\lambda_D} = \frac{\omega_{p,s}}{\omega_{g,s}}.$$

Nondimensionalization

Choose values for:

x_0	(space scale)	(e.g. ion skin depth δ_i),
m_0	(mass scale)	(e.g. ion mass m_i),
q_0	(charge scale)	(e.g. proton charge e),
T_0	(temperature),	(e.g. ion temperature T_i),
B_0	(magnetic field)	(e.g. $\omega_{g,i} m_i / e$), and
n_0	(number density)	(e.g. something $\gg 1/x_0^3$).

This implies typical values for:

$$
\begin{aligned}
v_0 &= \sqrt{T_0/m_0} && \text{(velocity scale)},\\
t_0 &= x_0/v_0 && \text{(time scale)},\\
E_0 &= B_0 v_0 && \text{(electric field)},\\
\rho_0 &= m_0 n_0 && \text{(mass density)},\\
\sigma_0 &= q_0 n_0 && \text{(charge density)},\\
J_0 &= q_0 n_0 v_0 && \text{(current density), and}\\
S_0 &= n_0 && \text{(no. pcls. per unit number density)}.
\end{aligned}
$$

Scale parameters are thus:

$$
\omega_{p,0}^2 := \frac{n_0 q_0^2}{\epsilon_0 m_0}, \qquad v_t := v_0, \qquad \lambda_D := \frac{v_t}{\omega_{p,0}},
$$

$$
\omega_g := \frac{q_0 B_0}{m_0}, \qquad v_A^2 := \frac{2 p_B}{\rho_0}, \qquad r_g := \frac{v_t}{\omega_g}, \qquad \delta_0 := \frac{v_A}{\omega_g}.
$$

Making the substitutions

$$
\begin{aligned}
t &= \hat{t} t_0, & \sigma &= \hat{\sigma}\sigma_0,\\
\mathbf{x} &= \hat{\mathbf{x}} x_0, & \mathbf{J} &= \hat{\mathbf{J}} J_0,\\
q &= \hat{q} q_0, & S_p(\mathbf{x}_p) &= \hat{S}_p(\hat{\mathbf{x}}_p) n_0,\\
m &= \hat{m} m_0, & c &= \hat{c} v_0,\\
n &= \hat{n} n_0, & \mathbf{v} &= \hat{\mathbf{v}} v_0,\\
\mathbf{B} &= \hat{\mathbf{B}} B_0, & \nabla &= x_0^{-1} \hat{\nabla}\\
\mathbf{E} &= \hat{\mathbf{E}} B_0 v_0, & &= x_0^{-1} \nabla_{\hat{\mathbf{x}}},
\end{aligned}
$$

in the fundamental equations gives an almost identical-appearing nondimensionalized system with only three nondimensional parameters:

1. $\hat{c} = \frac{c}{v_0}$ (light speed),
2. $\hat{\omega}_g := t_0 \omega_{g,0} = \frac{x_0}{r_g} =: \frac{1}{\hat{r}_g}$ (gyrofrequency or gyroradius),
3. $\hat{\lambda}_D := \frac{\lambda_D}{x_0} = \frac{1}{\hat{\omega}_p}$ (Debye length or plasma frequency).

Fundamental model: particle-Maxwell (nondimensionalized relativistic)

- **Maxwell's equations:**

$$\partial_{\hat{t}}\widehat{\mathbf{B}} + \widehat{\nabla} \times \widehat{\mathbf{E}} = 0,$$
$$\partial_{\hat{t}}\widehat{\mathbf{E}} - \widehat{c}^2\widehat{\nabla} \times \widehat{\mathbf{B}} = -\widehat{\mathbf{J}}/\widehat{\epsilon},$$
$$\widehat{\nabla} \cdot \widehat{\mathbf{B}} = 0, \quad \widehat{\nabla} \cdot \widehat{\mathbf{E}} = \widehat{\sigma}/\widehat{\epsilon}.$$

- **Charge moments:**

$$\widehat{\sigma} := \sum_p \widehat{S}_p(\widehat{\mathbf{x}}_p)\widehat{q}_p,$$
$$\widehat{\mathbf{J}} := \sum_p \widehat{S}_p(\widehat{\mathbf{x}}_p)\widehat{q}_p\widehat{\mathbf{v}}_p,$$

- **Particle equations:**

$$d_{\hat{t}}\widehat{\mathbf{x}}_p = \widehat{\mathbf{v}}_p,$$
$$d_{\hat{t}}(\gamma_p\widehat{\mathbf{v}}_p) = \widehat{\mathbf{a}}_{\mathrm{p}}(\widehat{\mathbf{x}}_p, \widehat{\mathbf{v}}_p),$$
$$\gamma_p^{-2} := 1 - (\widehat{\mathbf{v}}_p/\widehat{c})^2.$$

- **Lorentz acceleration**

$$\widehat{\mathbf{a}}_{\mathrm{p}}(\widehat{\mathbf{x}}, \widehat{\mathbf{v}}) = \widehat{\omega}_g \frac{\widehat{q}_p}{\widehat{m}_p}\left(\widehat{\mathbf{E}}(\widehat{\mathbf{x}}) + \widehat{\mathbf{v}} \times \widehat{\mathbf{B}}(\widehat{\mathbf{x}})\right)$$

Nondimensional parameters:

1. $\widehat{c}$: light speed.
2. $\widehat{\omega}_g = 1/\widehat{r}_g$: gyrofrequency or gyroradius.
3. $\widehat{\epsilon} = \widehat{\lambda}_D\left(\frac{\widehat{\lambda}_D}{\widehat{r}_g}\right) = \widehat{r}_g\left(\frac{\widehat{\lambda}_D}{\widehat{r}_g}\right)^2$: permittivity or Debye length.... (Note: $\frac{\widehat{\lambda}_D}{\widehat{r}_g} = \frac{\widehat{\omega}_g}{\widehat{\omega}_p} = \frac{v_A}{c}$.)

Matching with SI units:

- Acceleration is scaled by $\widehat{\omega}_g$.
- $\widehat{\omega}_g = 1$ if the time scale t_0 is chosen to be the gyroperiod.
- $\widehat{\omega}_g$ can be absorbed into the definition of $\widehat{\epsilon}$ or $\widehat{\lambda}_D$ along with charge or electromagnetic field.

Henceforth drop hats.

Problem: model based on particles is not a computationally accessible standard of truth (for space weather).

Solution: replace particles with a particle density function $f_s(t, \mathbf{x}, \gamma\mathbf{v})$ for each species s.

Accessible standard of truth: 2-species kinetic-Maxwell (relativistic)

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - \widehat{c}^2 \nabla \times \mathbf{B} = -\mathbf{J}/\widehat{\epsilon},$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\widehat{\epsilon}.$$

- **Charge moments:**

$$\sigma := \sum_s \frac{q_s}{m_s} \int f_s \, d(\gamma \mathbf{v}),$$
$$\mathbf{J} := \sum_s \frac{q_s}{m_s} \int \mathbf{v} f_s \, d(\gamma \mathbf{v}).$$

- **Kinetic equations:**

$$\partial_t f_i + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \mathbf{a}_i \cdot \nabla_{(\gamma \mathbf{v})} f_i = \mathcal{C}_i$$
$$\partial_t f_e + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e + \mathbf{a}_e \cdot \nabla_{(\gamma \mathbf{v})} f_e = \mathcal{C}_e$$

- **Lorentz acceleration:**

$$\mathbf{a}_i = \widehat{\omega}_g \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$
$$\mathbf{a}_e = \widehat{\omega}_g \frac{q_e}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

- **"Collision" operator**
  - includes all microscale effects
  - conservation: $\int \mathbf{m}(\mathcal{C}_i + \mathcal{C}_e)\, \gamma^{-1} \mathrm{d}(\gamma \mathbf{v}) = 0$, where $\mathbf{m} = (1, \gamma \mathbf{v}, \gamma)$.
  - decomposed as:

$$\mathcal{C}_i = \widetilde{\mathcal{C}}_{ii} + \overleftrightarrow{\mathcal{C}}_{ie},$$
$$\mathcal{C}_e = \widetilde{\mathcal{C}}_{ee} + \overleftrightarrow{\mathcal{C}}_{ei},$$

  where $\int \mathbf{m} \widetilde{\mathcal{C}}_{ss}\, \gamma^{-1} \mathrm{d}(\gamma \mathbf{v}) = 0$.

  - "collisionless": $\overleftrightarrow{\mathcal{C}}_{sp} \approx 0$.

- **BGK collision operator**

$$\widetilde{\mathcal{C}}_{ss} = \frac{\mathcal{M} - f}{\widehat{\tau}_{ss}},$$

where the entropy-maximizing distribution **M** shares physically conserved moments with f:

$$\mathcal{M} = \exp(\alpha \cdot \mathbf{m}),$$
$$\int \mathbf{m}(\mathcal{M} - f) \mathrm{d}(\gamma \mathbf{v}) = 0.$$

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - \widehat{c}^2 \nabla \times \mathbf{B} = -\mathbf{J}/\widehat{\epsilon},$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\widehat{\epsilon}.$$

- **Charge moments:**

$$\sigma := \sum_{\mathrm{s}} \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} \int f_{\mathrm{s}} \, d\mathbf{v},$$
$$\mathbf{J} := \sum_{\mathrm{s}} \frac{q_{\mathrm{s}}}{m_{\mathrm{s}}} \int \mathbf{v} f_{\mathrm{s}} \, d\mathbf{v}.$$

- **Kinetic equations:**

$$\boxed{\begin{aligned} \partial_t f_{\mathrm{i}} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \mathbf{a}_{\mathrm{i}} \cdot \nabla_{\mathbf{v}} f_{\mathrm{i}} &= \mathcal{C}_{\mathrm{i}} \\ \partial_t f_{\mathrm{e}} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e + \mathbf{a}_{\mathrm{e}} \cdot \nabla_{\mathbf{v}} f_{\mathrm{e}} &= \mathcal{C}_{\mathrm{e}} \end{aligned}}$$

- **Lorentz acceleration:**

$$\mathbf{a}_{\mathrm{i}} = \widehat{\omega}_g \frac{q_i}{m_i} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right),$$
$$\mathbf{a}_{\mathrm{e}} = \widehat{\omega}_g \frac{q_e}{m_e} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right).$$

- **"Collision" operator**
  - includes all microscale effects
  - conservation: $\int_{\mathbf{v}} \mathbf{m}(\mathcal{C}_{\mathrm{i}} + \mathcal{C}_{\mathrm{e}}) = 0$, where $\mathbf{m} = (1, \mathbf{v}, \|\mathbf{v}\|^2)$.
  - decomposed as:

$$\mathcal{C}_{\mathrm{i}} = \widetilde{\mathcal{C}}_{\mathrm{ii}} + \overleftrightarrow{\mathcal{C}}_{\mathrm{ie}},$$
$$\mathcal{C}_{\mathrm{e}} = \widetilde{\mathcal{C}}_{\mathrm{ee}} + \overleftrightarrow{\mathcal{C}}_{\mathrm{ei}},$$

  where $\int_{\mathbf{v}} \mathbf{m} \widetilde{\mathcal{C}}_{\mathrm{ii}} = 0 = \int_{\mathbf{v}} \mathbf{m} \widetilde{\mathcal{C}}_{\mathrm{ee}}$.

  - "collisionless": $\overleftrightarrow{\mathcal{C}}_{\mathrm{sp}} \approx 0$.

- **BGK collision operator**

$$\widetilde{\mathcal{C}}_{\mathrm{ss}} = \frac{\mathcal{M} - f}{\widehat{\tau}_{ss}},$$

where the Maxwellian distribution **M** shares physically conserved moments with f:

$$\mathcal{M} = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(\frac{-|\mathbf{c}|^2}{2\theta} \right),$$
$$\theta := \langle |\mathbf{c}|^2 / 2 \rangle.$$

Simplifications

Independent and dependent simplifications:

type	assumption	simplification	result
1.	$\widehat{r}_g \ll \lvert\nabla \log \mathbf{B}\rvert^{-1}$	gyro-average	gyro-kinetic/gyro-fluid
2.	$\widehat{\tau}_{ss}$ is small	take moments	fluid
3a.	$\widehat{\lambda}_D \to 0$	quasineutrality[1]	XMHD
3b.	$\widehat{c}$ is large	fast light	classical
3c.	$\widehat{c} \to \infty$ (implies $\widehat{\lambda}_D = 0$)[2]	Galileo	Ampere's law
3d.	$\widehat{r}_g \to 0$ (implies $\widehat{\lambda}_D = 0$)[3]	frozen flux	Ideal MHD[4]

…yield a commuting lattice of increasingly simplified models…

[1] $\sigma = \widehat{\lambda}_D \frac{\widehat{\lambda}_D}{\widehat{r}_g} \nabla \cdot \mathbf{E} \to 0$ as $\widehat{\lambda}_D \to 0$ if $\frac{\widehat{\lambda}_D}{\widehat{r}_g} = \frac{v_A}{c}$ is bounded.

[2] assuming $\widehat{c}\widehat{\lambda}_D = \frac{\widehat{c}}{\widehat{\omega}_p}$ is bounded

[3] assuming $\frac{\widehat{\lambda}_D}{\widehat{r}_g} = \frac{v_A}{c}$ is bounded

[4] Relativistic MHD assumes finite c and quasineutrality in the frame of reference of the plasma.

Model hierarchy

Kinetic two-species Maxwell	$\longrightarrow$	Kinetic two-species XMHD	$\longrightarrow$	Kinetic MHD
$\widehat{\tau}_{ss}$ small $\downarrow$		$\downarrow$		$\downarrow$
13-moment two-fluid Maxwell	$\longrightarrow$	13-moment two-fluid XMHD	$\longrightarrow$	13mom MHD
$\downarrow$		$\downarrow$		$\downarrow$
10-moment two-fluid Maxwell	$\longrightarrow$	10-moment two-fluid XMHD	$\longrightarrow$	10mom MHD
$\widehat{\tau}_{ss}\to 0$ $\downarrow$		$\downarrow$		$\downarrow$
5-moment two-fluid Maxwell	$\xrightarrow[\widehat{c}\to\infty]{}$	**5-moment two-fluid XMHD**	$\xrightarrow[\widehat{r}_g\to 0]{}$	**Ideal MHD**

Model hierarchy (gyro-averaged)

gyro-kinetic 2species Maxwell	⟶	gyro-kinetic 2species XMHD	⟶	gk MHD
↓		↓		↓
gyro-7-moment 2fluid Maxwell	⟶	gyro-7-moment 2fluid XMHD	⟶	g7m MHD
↓		↓		↓
gyro-6-moment 2fluid Maxwell	⟶	gyro-6-moment 2fluid XMHD	⟶	g6m MHD
↓		↓		↓
5-moment 2fluid Maxwell	⟶	**5-moment 2fluid XMHD**	⟶	**Ideal MHD**

Exactly gyro-averaged models fail to admit steady magnetic reconnection (for 2D problems rotationally symmetric about the origin)[5].

[5] and also fail to admit heat flux perpendicular to the magnetic field

Requirements for steady 2D symmetric magnetic reconnection

Consider the simplest reconnection scenario: *steady 2D reconnection symmetric under 180-degree rotation about the X-point.*

Theorem

Reconnection is impossible without viscosity or resistivity.

Argument:

- Rate of reconnection is the electric field strength at the X-point.
- Electric field strength at the X-point is resistive electric field plus viscous electric field.

Theorem (EAJ)

Reconnection is impossible for any conservative model for which heat flux is zero.

Argument:

- Steady reconnection requires entropy production near the X-point (via resistivity or viscosity).
- The X-point is a stagnation point.
- Without heat flux, heat accumulates at the X-point without bound.

Observation: in kinetic simulations, fast reconnection is supported by viscosity, not resistivity.

Conclusion: we need heat flux and viscosity in a fluid model of fast magnetic reconnection.

5-moment multi-fluid plasma

Evolution equations

$$\bar{\delta}_t \rho_s = 0$$
$$\rho_s d_t \mathbf{u}_s + \nabla p_s + \nabla \cdot \mathbb{P}_s^\circ = q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$$
$$\tfrac{3}{2} \bar{\delta}_t p_s + p_s \nabla \cdot \mathbf{u}_s + \mathbb{P}_s^\circ : \nabla \mathbf{u}_s + \nabla \cdot \mathbf{q}_s = 0$$

Evolved moments

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \frac{3}{2} p_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2} \|\mathbf{c}_s\|^2 \end{bmatrix} f_s \, d\mathbf{v}$$

Definitions

$$\bar{\delta}_t(\alpha) := \partial_t \alpha + \nabla \cdot (\mathbf{u}_s \alpha)$$
$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s$$
$$n_s := \rho_s / m_s$$

Relaxation (diffusive) flux closures:

$$\mathbb{P}_s^\circ = \int (\mathbf{c}_s \mathbf{c}_s - \|\mathbf{c}_s\|^2 \mathbb{I}/3) \, f_s \, d\mathbf{v}$$
$$= -2\boldsymbol{\mu} : \boldsymbol{e}^\circ,$$
$$\mathbf{q}_s = \int \tfrac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \, f_s \, d\mathbf{v}$$
$$= -\mathbf{k} \cdot \nabla T.$$

10-moment multi-fluid plasma

Evolution equations

$$\bar{\delta}_t \rho_s = 0$$
$$\rho_s d_t \mathbf{u}_s + \nabla\cdot\mathbb{P}_s = q_s n_s(\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$$
$$\bar{\delta}_t \mathbb{P}_s + \mathrm{Sym2}(\mathbb{P}_s\cdot\nabla\mathbf{u}_s) + \nabla\cdot\underline{\underline{q}}_s = q_s n_s\,\mathrm{Sym2}(\mathbb{T}_s\times\mathbf{B}) + \mathbb{R}_s$$

Evolved moments

$$\begin{bmatrix}\rho_s\\ \rho_s\mathbf{u}_s\\ \mathbb{P}_s\end{bmatrix} = \int \begin{bmatrix}1\\ \mathbf{v}\\ \mathbf{c}_s\mathbf{c}_s\end{bmatrix} f_s\, d\mathbf{v}$$

Definitions

$$\bar{\delta}_t(\alpha) := \partial_t\alpha + \nabla\cdot(\mathbf{u}_s\alpha)$$
$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s$$
$$\mathrm{Sym2}(A) := A + A^T$$
$$n_s := \rho_s/m_s$$
$$\mathbb{T} := \mathbb{P}/n$$

Relaxation source term closures:

$$\mathbb{R}_s = \int \mathbf{c}_s\mathbf{c}_s\, C_s\, d\mathbf{v} = -\mathbb{P}_s^\circ/\tau,$$

Relaxation (diffusive) flux closures:

$$\underline{\underline{q}}_s = \int \mathbf{c}_s\mathbf{c}_s\mathbf{c}_s\, f_s\, d\mathbf{c}_s$$
$$= -\tfrac{2}{5}\,\mathrm{Sym}\left(k_s\widetilde{\mathbf{K}}_s : \mathrm{Sym3}\left(\frac{\mathbb{T}_s}{T_s}\cdot\nabla\mathbb{T}_s\right)\right)$$

Evolution equations

$$\bar{\delta}_t \rho_s = 0$$

$$\rho_s d_t \mathbf{u}_s + \nabla \cdot \mathbb{P}_s = q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$$

$$\bar{\delta}_t \mathbb{P}_s + \mathrm{Sym2}(\mathbb{P}_s \cdot \nabla \mathbf{u}_s) + \nabla \cdot \underline{\underline{\underline{q}}}_s = q_s n_s \,\mathrm{Sym2}(\mathbb{T}_s \times \mathbf{B}) + \mathbb{R}_s$$

$$\bar{\delta}_t \mathbf{q}_s + \mathbf{q}_s \cdot \nabla \mathbf{u}_s + \underline{\underline{\underline{q}}}_s : \nabla \mathbf{u}_s + \mathbb{P}_s : \nabla \underline{\underline{\Theta}}_s + \mathbb{P}_s \cdot \nabla \theta_s + \nabla \cdot \underline{\underline{R}}_s = \mathrm{Sym3}(\mathbb{P}_s \mathbb{P}_s)/\rho_s + \frac{q_s}{m_s} \underline{\underline{\underline{q}}}_s \times \mathbf{B} + \widetilde{\mathbf{q}}_{ss,t}$$

Evolved moments

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \mathbb{P}_s \\ \mathbf{q}_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \mathbf{c}_s \mathbf{c}_s \\ \frac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} f_s \, d\mathbf{v}$$

Definitions

$$\bar{\delta}_t(\alpha) := \partial_t \alpha + \nabla \cdot (\mathbf{u}_s \alpha)$$

$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s, \qquad \mathrm{Sym2}(A) := A + A^T,$$

$$n_s := \rho_s / m_s, \qquad \mathbb{T} := \mathbb{P}/n,$$

$$\underline{\underline{\Theta}} := \mathbb{P}/\rho, \qquad \theta := \mathrm{tr}\, \underline{\underline{\Theta}}/2,$$

Relaxation source term closures:

$$\begin{bmatrix} \mathbb{R}_s \\ \widetilde{\mathbf{q}}_{ss,t} \end{bmatrix} = \int \begin{bmatrix} \mathbf{c}_s \mathbf{c}_s \\ \frac{1}{2} \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} C_{ss} \, d\mathbf{v} = \frac{-1}{\tau_s} \begin{bmatrix} \mathbb{P}_s^\circ \\ \mathrm{Pr}\, \mathbf{q}_s \end{bmatrix}$$

Hyperbolic flux closures:

$$\begin{bmatrix} \underline{\underline{\underline{q}}}_s \\ \underline{\underline{R}}_s \end{bmatrix} = \int \begin{bmatrix} \mathbf{c}_s \mathbf{c}_s \mathbf{c}_s \\ \mathbf{c}_s \mathbf{c}_s \|\mathbf{c}_s\|^2 \end{bmatrix} f_s(\mathbf{c}_s) \, d\mathbf{c}_s$$

Relaxation closures for viscosity and heat flux

5-moment:

$$\mathbb{P}^\circ = -2\mu\widetilde{\boldsymbol{\mu}} : \boldsymbol{e}^\circ$$

$$\mathbf{q} = -k\widetilde{\mathbf{k}} \cdot \nabla T$$

10-moment:

$$\partial_t \mathbb{P}^\circ = -\mathbb{P}^\circ/\tau \ldots$$

$$\underline{\underline{\underline{q}}} = -\tfrac{2}{5}\,\mathrm{Sym}\left(k\widetilde{\mathbf{K}} \vdots \mathrm{Sym3}\left(\frac{\mathbb{T}}{T} \cdot \nabla \mathbb{T}\right)\right)$$

13-moment:

$$\partial_t \mathbb{P}^\circ = -\mathbb{P}^\circ/\tau \ldots$$

$$\partial_t \mathbf{q} = -\mathbf{q}/\widetilde{\tau} \ldots$$

Relaxation periods:

$$\tau \sim \tau_0 \sqrt{mT^3}/n,$$

$$\widetilde{\tau} := \tau / \,\mathrm{Pr},$$

$$\tau_0 := \frac{12\pi^{3/2}}{\ln\Lambda}\left(\frac{\epsilon_0}{e^2}\right)^2.$$

Diffusion parameters:

$$\mu = \tau p$$

$$\tfrac{2}{5}k = \frac{\mu}{m\,\mathrm{Pr}}$$

Diffusion coefficients:

$$\widetilde{\mathbf{k}} = \mathbb{I}_\parallel + \tfrac{1}{1+\widetilde{\varpi}^2}(\mathbb{I}_\perp - \widetilde{\varpi}\mathbb{I}_\wedge),$$

$$\widetilde{\boldsymbol{\mu}} = \tfrac{1}{2}(3\mathbb{I}_\parallel^2 + \mathbb{I}_\perp^2) + \tfrac{2}{1+\varpi^2}(\mathbb{I}_\perp\mathbb{I}_\parallel - \varpi\mathbb{I}_\wedge\mathbb{I}_\parallel)$$
$$+ \tfrac{1}{1+4\varpi^2}(\tfrac{1}{2}(\mathbb{I}_\perp^2 - \mathbb{I}_\wedge^2) - 2\varpi\mathbb{I}_\wedge\mathbb{I}_\perp),$$

$$\widetilde{\mathbf{K}} = \left(\mathbb{I}_\parallel^3 + \tfrac{3}{2}\mathbb{I}_\parallel(\mathbb{I}_\perp^2 + \mathbb{I}_\wedge^2)\right)$$
$$+ \tfrac{3}{1+\widetilde{\varpi}^2}\left(\mathbb{I}_\perp\mathbb{I}_\parallel^2 - \widetilde{\varpi}\mathbb{I}_\wedge\mathbb{I}_\parallel^2\right)$$
$$+ \tfrac{3}{1+4\widetilde{\varpi}^2}\left(\tfrac{1}{2}(\mathbb{I}_\perp^2 - \mathbb{I}_\wedge^2)\mathbb{I}_\parallel - 2\widetilde{\varpi}\mathbb{I}_\wedge\mathbb{I}_\perp\mathbb{I}_\parallel\right)$$
$$+ (k_0\mathbb{I}_\perp^3 + k_1\mathbb{I}_\wedge\mathbb{I}_\perp^2 + k_2\mathbb{I}_\wedge^2\mathbb{I}_\perp + k_3\mathbb{I}_\wedge^3),$$

$$k_3 := \frac{-6\widetilde{\varpi}^3}{1 + 10\widetilde{\varpi}^2 + 9\widetilde{\varpi}^4},$$

$$k_2 := \frac{6\widetilde{\varpi}^2 + 3\widetilde{\varpi}(1 + 3\widetilde{\varpi}^2)k_3}{1 + 7\widetilde{\varpi}^2},$$

$$k_1 := \frac{-3\widetilde{\varpi} + 2\widetilde{\varpi}k_2}{1 + 3\widetilde{\varpi}^2},$$

$$k_0 := 1 + \widetilde{\varpi}k_1.$$

Definitions:

$$\mathrm{Pr} := \text{Prandtl no.}$$
$$(\text{e.g. } \tfrac{2}{3} \text{ or } 1),$$
$$\varpi := \tau\frac{q}{m}|\mathbf{B}|,$$
$$\widetilde{\varpi} := \widetilde{\tau}\frac{q}{m}|\mathbf{B}|,$$
$$\mathbf{b} := \mathbf{B}/|\mathbf{B}|,$$
$$\mathbb{I} := \text{identity},$$
$$\mathbb{I}_\parallel := \mathbf{bb},$$
$$\mathbb{I}_\perp := \mathbb{I} - \mathbf{bb},$$
$$\mathbb{I}_\wedge := \mathbf{b} \times \mathbb{I}.$$

In this frame the species index s is suppressed. All products of even-order tensors are *splice products* satisfying

$$(AB)_{j_1 j_2 k_1 k_2} := A_{j_1 k_1} B_{j_2 k_2},$$
$$(ABC)_{j_1 j_2 j_3 k_1 k_2 k_3} := A_{j_1 k_1} B_{j_2 k_2} C_{j_3 k_3}.$$

Conclusion

Available closures with nonzero heat flux:

1. For 5-moment and 10-moment gas dynamics, physical heat flux closures are *diffusive* and dependent on magnetic field:
   - diffusive closures are time-averaged and must incorporate the rotational effects of the magnetic field
   - heat conductivity is highly anisotropic in the presence of a strong guiding magnetic field.
     - tokamak: parallel heat conductivity can be a million times stronger than perpendicular heat conductivity.
     - need to avoid anomalous cross-field diffusion.
     - need high-order accuracy or field-aligned coordinates.
2. if heat flux and viscosity are evolved, a simple relaxation closure can be used:
   - 13-moment system: can use Grad's explicit closure with hyperbolicity limiting.
   - 14-moment entropy-maximizing system evolves $\int |\mathbf{c}_s|^4 f_s d\mathbf{c}_s$. (Maturing; see work by McDonald, Torrilhon, Groth.)

Outline

Models which evolve Maxwell's equations and classical gas dynamics fail to satisfy a relativity principle. Magnetohydrodyamics (MHD) remedies this problem by assuming that the light speed is infinite. Then Maxwell's equations simplify to

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \cancel{c^{-2}\partial_t \mathbf{E}}, \qquad \mu_0 \sigma = 0 + \cancel{c^{-2}\nabla \cdot \mathbf{E}}$$

This system is Galilean-invariant, but its relationship to gas-dynamics is fundamentally different:

variable	**MHD**	**2-fluid-Maxwell**
E	supplied by *Ohm's law* (from gas dynamics)	evolved (from **B** and **J**)
J	$\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ (comes from **B**)	$\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$ (from gas dynamics)
σ	$\sigma = 0$ (quasineutrality) (gas-dynamic constraint)	$\sigma = e(n_i - n_e)$ (electric field constraint)

MHD: charge neutrality

The assumption of charge neutrality reduces the number of gas-dynamic equations that must be solved:

- ***net* density** evolution

  The density of each species is the same:

  $$n_\mathrm{i} = n_\mathrm{e} = n$$

- ***net* velocity** evolution

  The species fluid velocities can be inferred from the net current, net velocity, and density:

  $$\mathbf{u}_\mathrm{i} = \mathbf{u} + \frac{m_\mathrm{e}}{m_\mathrm{i} + m_\mathrm{e}} \frac{\mathbf{J}}{ne}, \qquad \mathbf{u}_\mathrm{e} = \mathbf{u} - \frac{m_\mathrm{i}}{m_\mathrm{i} + m_\mathrm{e}} \frac{\mathbf{J}}{ne}.$$

For each species $\mathrm{s} \in \{\mathrm{i}, \mathrm{e}\}$, rescaling momentum evolution by $\mathbf{q}_\mathrm{s}/m_\mathrm{s}$ gives the current evolution equation

$$\partial_t \mathbf{J}_\mathrm{s} + \nabla\cdot(\mathbf{u}_\mathrm{s}\mathbf{J}_\mathrm{s} + (q_\mathrm{s}/m_\mathrm{s})\mathbb{P}_\mathrm{s}) = (q_\mathrm{s}^2/m_\mathrm{s})n(\mathbf{E} + \mathbf{u}_\mathrm{s}\times\mathbf{B}) + (q_\mathrm{s}/m_\mathrm{s})\mathbf{R}_\mathrm{s}.$$

Summing over both species and using charge neutrality gives net current evolution:

$$\partial_t \mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{JJ}\right) + e\nabla\cdot\left(\frac{\mathbb{P}_\mathrm{i}}{\mathbf{m}_\mathrm{i}} - \frac{\mathbb{P}_\mathrm{e}}{\mathbf{m}_\mathrm{e}}\right) = \frac{e^2\rho}{m_\mathrm{i}m_\mathrm{e}}\left(\mathbf{E} + \left(\mathbf{u} - \frac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{J}\right)\times\mathbf{B} - \frac{\mathbf{R}_\mathrm{e}}{en}\right).$$

A closure for the collisional term is $\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta}\cdot\mathbf{J} + \boldsymbol{\beta}_\mathrm{e}\cdot\mathbf{q}_\mathrm{e}$. (Note: $\eta \sim \frac{m_\mathrm{red}}{e^2 n\tau_\mathrm{slow}}$).

Ohm's law is current evolution solved for the electric field:

$$\begin{aligned}
\mathbf{E} =& \mathbf{B}\times\mathbf{u} && \text{(ideal term)}\\
&+ \tfrac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{J}\times\mathbf{B} && \text{(Hall term)}\\
&+ \boldsymbol{\eta}\cdot\mathbf{J} && \text{(resistive term)}\\
&+ \boldsymbol{\beta}_\mathrm{e}\cdot\mathbf{q}_\mathrm{e} && \text{(thermoelectric term)}\\
&+ \tfrac{1}{e\rho}\nabla\cdot(m_\mathrm{e}\mathbb{P}_\mathrm{i} - m_\mathrm{i}\mathbb{P}_\mathrm{e}) && \text{(pressure term)}\\
&+ \tfrac{m_\mathrm{i}m_\mathrm{e}}{e^2\rho}\left[\partial_t\mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \tfrac{m_\mathrm{i}-m_\mathrm{e}}{e\rho}\mathbf{JJ}\right)\right] && \text{(inertial term).}
\end{aligned}$$

Since $\mathbf{J} = \mu_0^{-1}\nabla\times\mathbf{B}$, Ohm's law gives an implicit closure to the induction equation, $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$ (so retaining the inertial term entails an implicit numerical method).

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}_\mathrm{e}}{en} + \mathbf{B} \times \mathbf{u} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B}$$

$$+ \frac{1}{e\rho} \nabla \cdot \left(m_\mathrm{e}(p_\mathrm{i}\mathbb{I} + \mathbb{P}_\mathrm{i}^\circ) - m_\mathrm{i}(p_\mathrm{e}\mathbb{I} + \mathbb{P}_\mathrm{e}^\circ) \right)$$

$$+ \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right]$$

Pressure evolution

$$\tfrac{3}{2} n d_t T_\mathrm{i} + p_\mathrm{i} \nabla \cdot \mathbf{u}_\mathrm{i} + \mathbb{P}^\circ{}_\mathrm{i} : \nabla \mathbf{u}_\mathrm{i} + \nabla \cdot \mathbf{q}_\mathrm{i} = Q_\mathrm{i},$$

$$\tfrac{3}{2} n d_t T_\mathrm{e} + p_\mathrm{e} \nabla \cdot \mathbf{u}_\mathrm{e} + \mathbb{P}^\circ{}_\mathrm{e} : \nabla \mathbf{u}_\mathrm{e} + \nabla \cdot \mathbf{q}_\mathrm{e} = Q_\mathrm{e};$$

Definitions:

$$d_t := \partial_t + \mathbf{u}_\mathrm{s} \cdot \nabla,$$

$$\mathbb{P}^\mathrm{d} := m_\mathrm{red} n \mathbf{ww},$$

$$\mathbf{w} = \frac{\mathbf{J}}{en}, \quad m_\mathrm{red}^{-1} := m_\mathrm{e}{}^{-1} + m_\mathrm{i}{}^{-1}.$$

Closures:

$$\mathbb{P}_\mathrm{s}^\circ = -2\mu : (\nabla \mathbf{u})^\circ$$

$$\mathbf{q}_\mathrm{s} = -k \cdot \nabla T$$

$$\frac{\mathbf{R}_\mathrm{e}}{en} = \eta \cdot \mathbf{J} + \beta_\mathrm{e} \cdot \mathbf{q}_\mathrm{e}$$

$$Q_\mathrm{s} = ?$$

Equations of 10-moment 2-fluid MHD

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}_\mathrm{e}}{en} + \mathbf{B} \times \mathbf{u} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B} + \frac{1}{e\rho} \nabla \cdot (m_\mathrm{e}\mathbb{P}_\mathrm{i} - m_\mathrm{i}\mathbb{P}_\mathrm{e}) + \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{JJ} \right) \right]$$

Definitions:

$$d_t := \partial_t + \mathbf{u}_\mathrm{s} \cdot \nabla,$$

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$$\mathbf{w} = \frac{\mathbf{J}}{en}, \quad m_\mathrm{red}^{-1} := m_\mathrm{e}{}^{-1} + m_\mathrm{i}{}^{-1}.$$

Closures:

$$\mathbb{R}_\mathrm{s} = -\frac{1}{\tau} \mathbb{P}_\mathrm{s}^\circ$$

$$\underline{\underline{\underline{q_\mathrm{s}}}} = -\frac{2}{5} \mathbf{K}_\mathrm{s} \vdots \mathrm{Sym3}\left(\frac{\mathbb{T}_\mathrm{s}}{T_\mathrm{s}} \cdot \nabla \mathbb{T}_\mathrm{s} \right)$$

$$\frac{\mathbf{R}_\mathrm{e}}{en} = \boldsymbol{\eta} \cdot \mathbf{J} + \boldsymbol{\beta}_\mathrm{e} \cdot \mathbf{q}_\mathrm{e}$$

$$\mathbb{Q}_\mathrm{s} = ?$$

Pressure evolution

$$n_\mathrm{i} d_t \mathbb{T}_\mathrm{i} + \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \cdot \nabla \mathbf{u}_\mathrm{i}) + \nabla \cdot \underline{\underline{\underline{q_\mathrm{i}}}} = \frac{q_\mathrm{i}}{m_\mathrm{i}} \mathrm{Sym2}(\mathbb{P}_\mathrm{i} \times \mathbf{B}) + \mathbb{R}_\mathrm{i} + \mathbb{Q}_\mathrm{i},$$

$$n_\mathrm{e} d_t \mathbb{T}_\mathrm{e} + \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \cdot \nabla \mathbf{u}_\mathrm{e}) + \nabla \cdot \underline{\underline{\underline{q_\mathrm{e}}}} = \frac{q_\mathrm{e}}{m_\mathrm{e}} \mathrm{Sym2}(\mathbb{P}_\mathrm{e} \times \mathbf{B}) + \mathbb{R}_\mathrm{e} + \mathbb{Q}_\mathrm{e}$$

Equations of 13-moment 2-fluid MHD

mass and momentum:

$$\partial_t \rho + \nabla\cdot(\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla\cdot(\mathbb{P}_{\rm i} + \mathbb{P}_{\rm e} + \mathbb{P}^{\rm d}) = \mathbf{J}\times\mathbf{B}$$

Electromagnetism

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0,$$

$$\mathbf{J} = \mu_0^{-1}\nabla\times\mathbf{B}$$

Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}_{\rm e}}{en} + \mathbf{B}\times\mathbf{u} + \frac{m_{\rm i}-m_{\rm e}}{e\rho}\mathbf{J}\times\mathbf{B} + \frac{1}{e\rho}\nabla\cdot(m_{\rm e}\mathbb{P}_{\rm i} - m_{\rm i}\mathbb{P}_{\rm e}) + \frac{m_{\rm i}m_{\rm e}}{e^2\rho}\left[\partial_t\mathbf{J} + \nabla\cdot\left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_{\rm i}-m_{\rm e}}{e\rho}JJ\right)\right]$$

Diffusive relaxation closures:

$$\frac{\mathbf{R}_{\rm e}}{en} = \boldsymbol{\eta}\cdot\mathbf{J} + \boldsymbol{\beta}_{\rm e}\cdot\mathbf{q}_{\rm e}$$

Relaxation source term closures:

$$\mathbb{R}_{\rm s} = -\mathbb{P}_{\rm s}^{\circ}/\tau_{\rm s}$$

$$\widetilde{\mathbf{q}}_{{\rm ss},t} = -\mathbf{q}_{\rm s}/\widetilde{\tau}_{\rm s}$$

Hyperbolic flux closures:

$$\begin{bmatrix} \underline{\underline{q}}_{\rm s} \\ \underline{\underline{R}}_{\rm s} \end{bmatrix} = \int \begin{bmatrix} \mathbf{c}_{\rm s}\mathbf{c}_{\rm s}\mathbf{c}_{\rm s} \\ \mathbf{c}_{\rm s}\mathbf{c}_{\rm s}\|\mathbf{c}_{\rm s}\|^2 \end{bmatrix} f_{\rm s}(\mathbf{c}_{\rm s})\, d\mathbf{c}_{\rm s}$$

Interspecies forcing closures:

$$\begin{bmatrix} \mathbf{R}_{\rm s} \\ \mathbb{Q}_{\rm s} \\ \overleftrightarrow{\mathbf{q}}_{{\rm s},t} \end{bmatrix} = ?$$

Pressure evolution

$$n_{\rm i} d_t \mathbb{T}_{\rm i} + \mathrm{Sym2}(\mathbb{P}_{\rm i}\cdot\nabla\mathbf{u}_{\rm i}) + \nabla\cdot\underline{\underline{q}}_{\rm i} = \frac{q_{\rm i}}{m_{\rm i}}\mathrm{Sym2}(\mathbb{P}_{\rm i}\times\mathbf{B}) + \mathbb{R}_{\rm i} + \mathbb{Q}_{\rm i},$$

$$n_{\rm e} d_t \mathbb{T}_{\rm e} + \mathrm{Sym2}(\mathbb{P}_{\rm e}\cdot\nabla\mathbf{u}_{\rm e}) + \nabla\cdot\underline{\underline{q}}_{\rm e} = \frac{q_{\rm e}}{m_{\rm e}}\mathrm{Sym2}(\mathbb{P}_{\rm e}\times\mathbf{B}) + \mathbb{R}_{\rm e} + \mathbb{Q}_{\rm e}$$

Heat flux evolution

$$\bar{\delta}_t\mathbf{q}_{\rm s} + \mathbf{q}_{\rm s}\cdot\nabla\mathbf{u}_{\rm s} + \underline{\underline{q}}_{\rm s} : \nabla\mathbf{u}_{\rm s} + \mathbb{P}_{\rm s} : \nabla\underline{\underline{\Theta}}_{\rm s} + \frac{3}{2}\mathbb{P}_{\rm s}\cdot\nabla\theta_{\rm s} + \frac{1}{2}\nabla\cdot\underline{\underline{R}}_{\rm s} = \rho(3\theta\Theta + 2\Theta\cdot\Theta) + \frac{q_{\rm s}}{m_{\rm s}}\underline{\underline{q}}_{\rm s}\times\mathbf{B} + \widetilde{\mathbf{q}}_{{\rm ss},t} + \overleftrightarrow{\mathbf{q}}_{{\rm s},t}$$

Asymptotic-preserving hierarchy

For efficient multiscale simulation, discretizations need to be *asymptotic-preserving (AP)* with respect to simpler models:

$$\begin{array}{ccc} \text{micro-scheme}(\epsilon, h) & \xrightarrow{h\to 0} & \text{micro-physics}(\epsilon) \\ \downarrow{\scriptstyle \epsilon\to 0} & & \downarrow{\scriptstyle \epsilon\to 0} \\ \textbf{macro-scheme}(h) & \xrightarrow{h\to 0} & \text{macro-physics} \end{array}$$

- Micro-scheme is ***AP*** with respect to macro-physics if macro-scheme exists and is stable.
- In macro-physics limit, micro-physics waves disappear — either are damped (strong limit) or become infinitely fast/fine (weak limit).

Keys to designing good AP schemes:

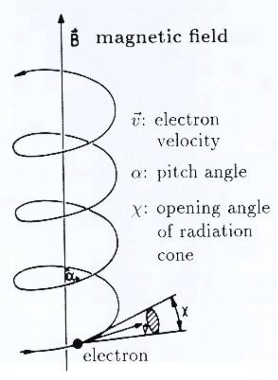
- Thoroughly *understand the limit* (fit the framework).
- Use an *implicit scheme* to skip over unresolved microphysics (e.g. to damp fast waves).
- *Preserve invariants* (mimic the physics with conforming discretizations:

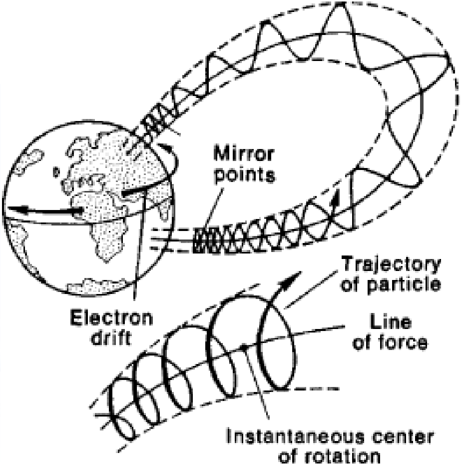
$$\begin{array}{ccc} \text{scheme} & \xrightarrow{h\to 0} & \text{physics} \\ \downarrow & & \downarrow \\ \text{[disc. op. A]} & \xrightarrow{h\to 0} & \text{[op. A]} \\ \downarrow & & \downarrow \\ \text{[disc. op. B]} & \xrightarrow{h\to 0} & \text{[op. B]} \end{array}$$

Outline

"Frozen-in" magnetic field lines

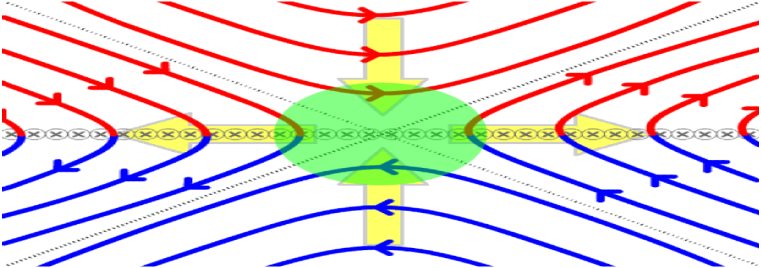
- charged particles spiral around magnetic field lines.
- viewed from a distance, the particles are stuck to the field lines.
- so magnetic field lines approximately move with the plasma.





Magnetic Reconnection

- **Start:** oppositely directed field lines are driven towards each other.
- Field lines reconnect at the **X-point**.
- **Lower energy state:** change topology of field lines
- Results in large energy release in the form of oppositely directed jets



2D separator reconnection

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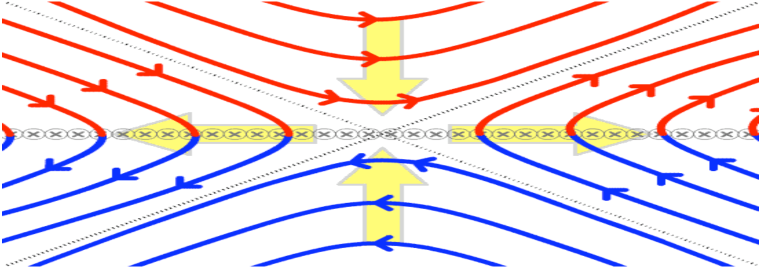

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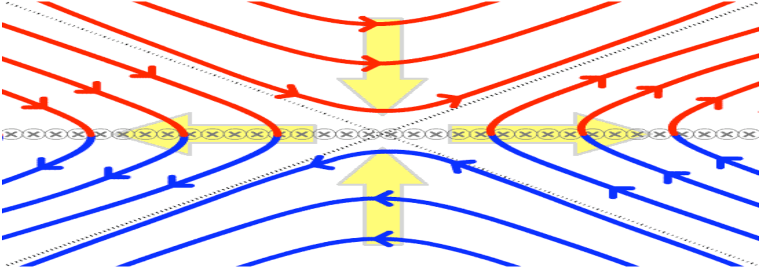

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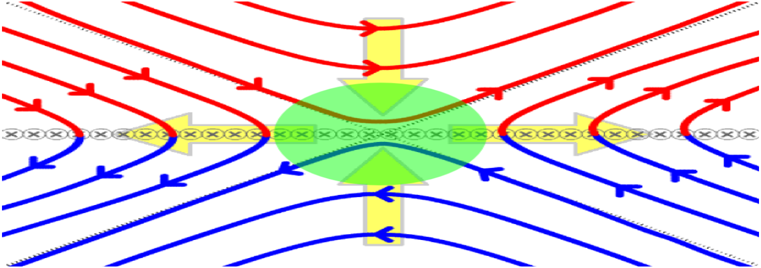

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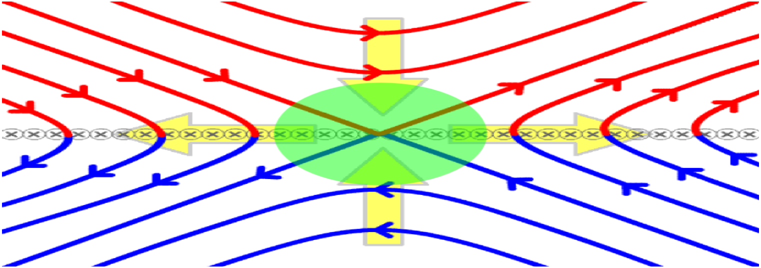

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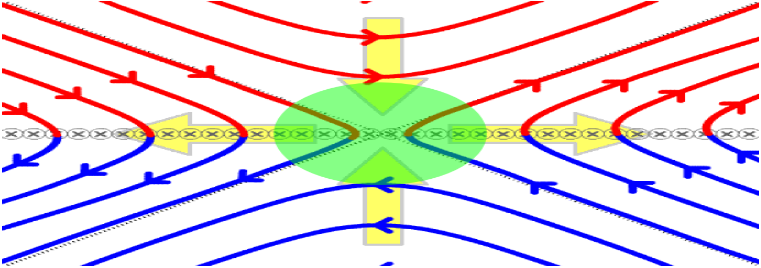

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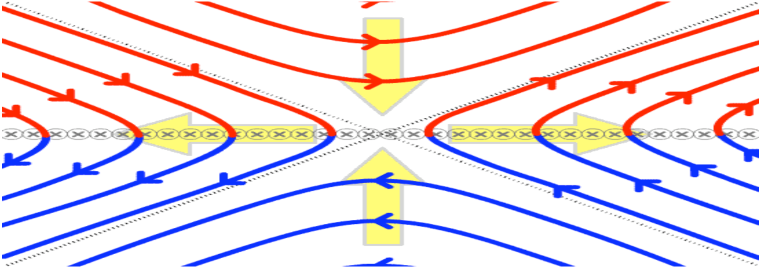

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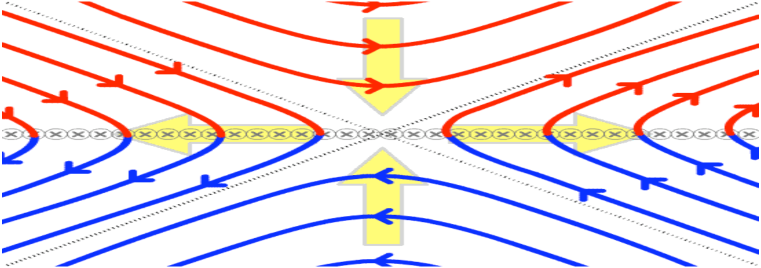

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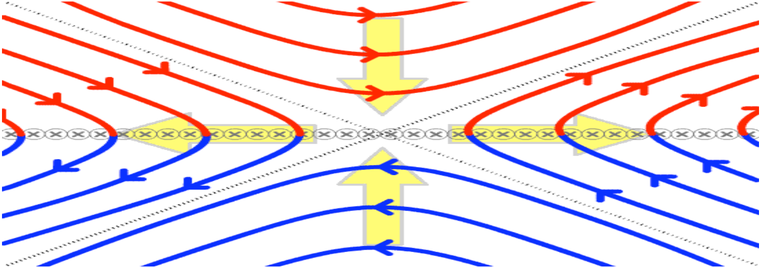


2D separator reconnection

Simplest plasma model: Ideal/Hall MHD

In Ideal/Hall MHD Faraday's law is of the form

$$\partial_t \mathbf{B} + \nabla \times \underbrace{(\mathbf{B} \times \mathbf{u}_c)}_{\text{ideal } \mathbf{E}} = 0,$$

which says that $\mathbf{u}_c$ is a ***flux-transporting flow*** for **B**.

So magnetic reconnection is not possible in Ideal/Hall MHD.

X-point analysis

At the X-point the momentum equation ("Ohm's law") reduces to

$$\text{rate of reconnection} = \mathbf{E}_3(0) = \frac{-\mathbf{R}_\mathrm{i}}{e n_\mathrm{i}} \qquad \text{(resistive term)}$$

$$+ \frac{\nabla \cdot \mathbb{P}_\mathrm{i}}{e n_\mathrm{i}} \qquad \text{(pressure term)}$$

$$+ \frac{m_\mathrm{i}}{e} \partial_t \mathbf{u}_\mathrm{i} \qquad \text{(inertial term)}$$

Consequences:

1. *Collisionless reconnection* is supported by the inertial or pressure term.
2. For the 5-moment model the *inertial* term must support the reconnection; i.e. each species velocity at the origin should track exactly with reconnected flux.
3. For *steady-state* reconnection without resistivity the *pressure* term must provide for the reconnection.

resistive MHD

Resistive Ohm's law allows reconnection:

$$\mathbf{E} = \mathbf{B} \times \mathbf{u}_c + \eta \cdot \mathbf{J}.$$

- Magnetic reconnection is slow with uniform resistivity.
- Anomalous resistivity allows desired rate of reconnection.
  - But a formula for anomalous resistivity has been elusive.
  - Fast reconnection is not supported by physical anomalous resistivity (supported instead by anomalous viscosity).
  - Hall MHD plus small resistivity allows fast reconnection (but supported by the wrong term in Ohm's law).

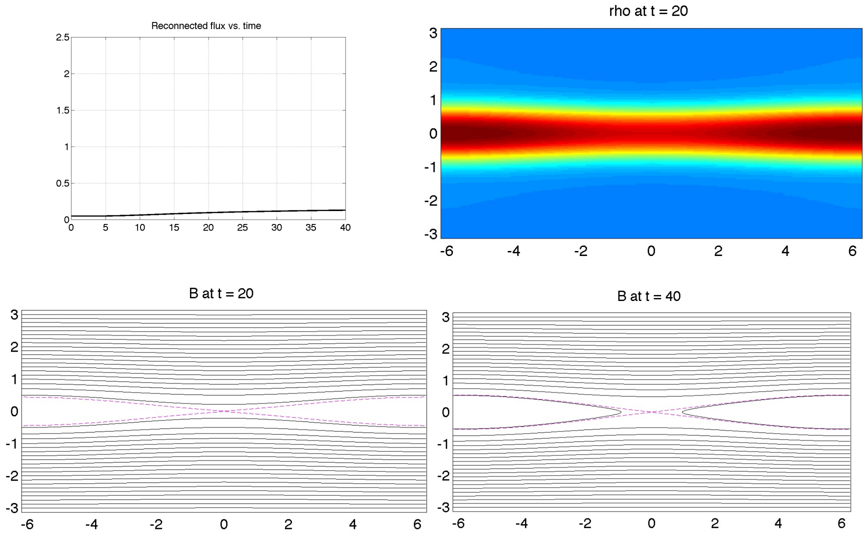
Collisionless models

Observation: Fast reconnection always occurs on collisionless or weakly collisional spatial scales.

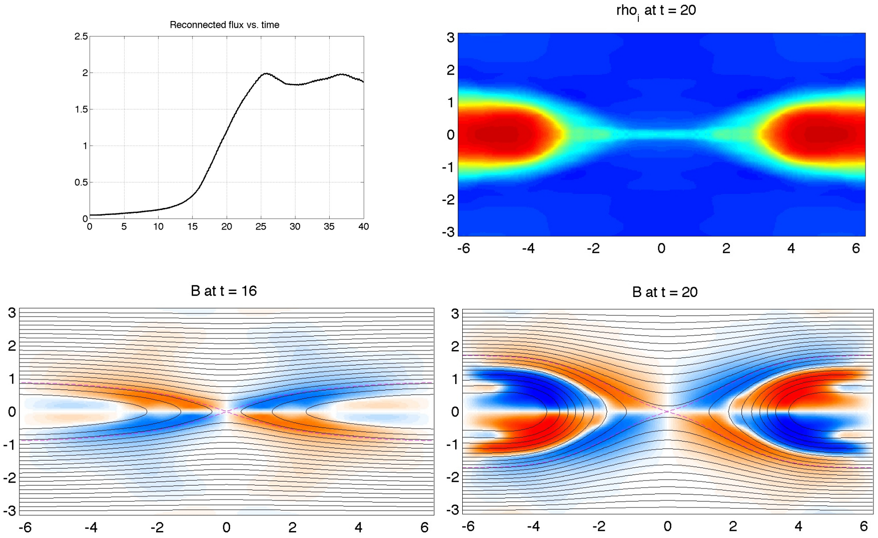
Why: Mean free path $\gg$ reconnection scale.

Therefore in model ignore interspecies collision operator (so zero resistivity).

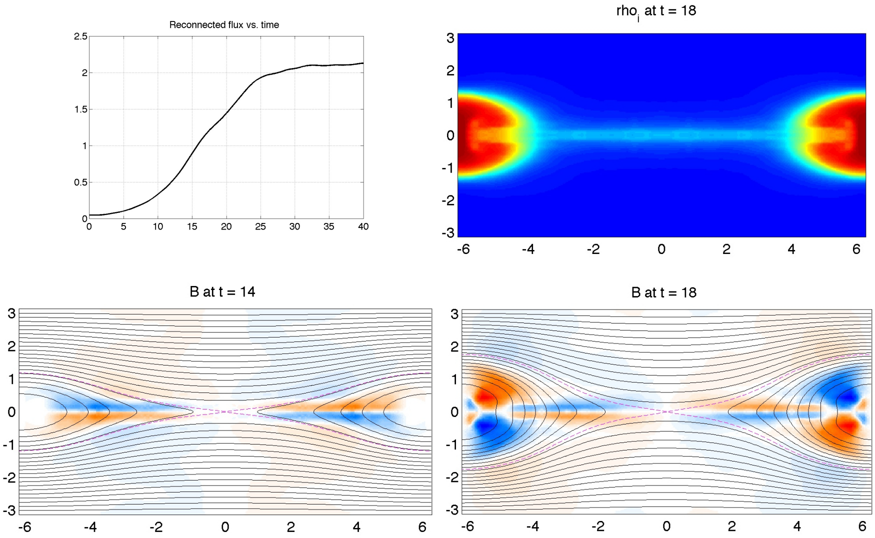
GEM: Resistive MHD ($\eta = 5 \times 10^{-3}$)



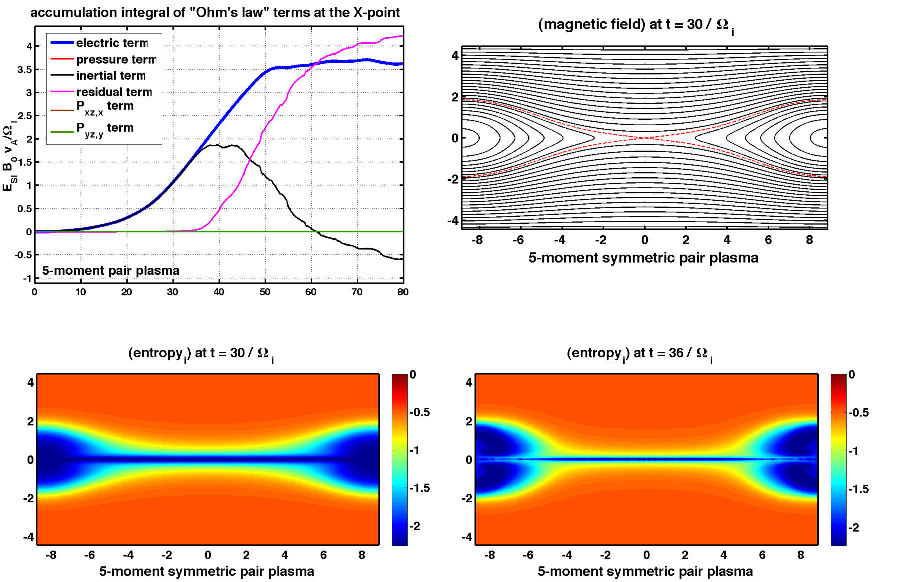
GEM: 5-moment 2-fluid-Maxwell: $\frac{m_i}{m_e} = 25$



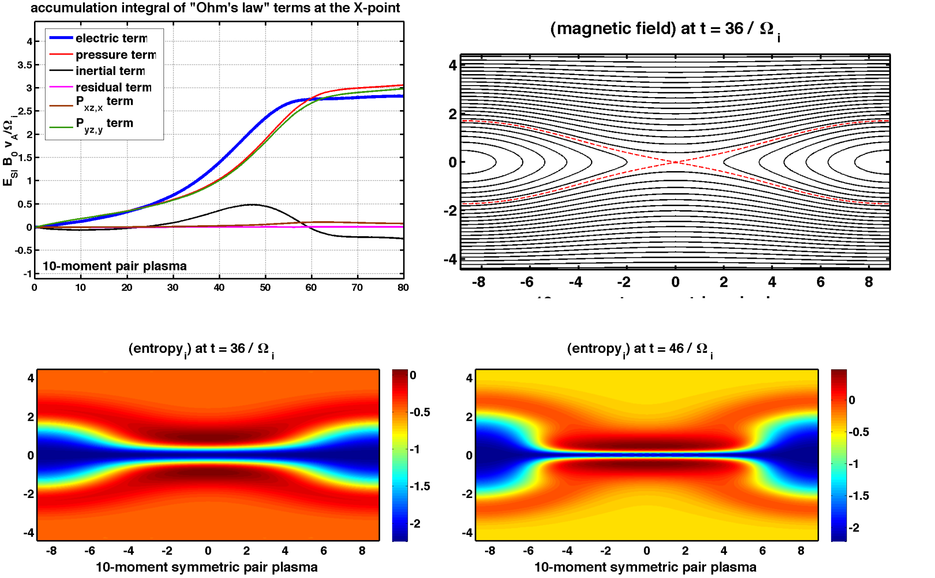
GEM: 5-moment 2-fluid Maxwell: $\frac{m_i}{m_e} = 1$ (no Hall term) coarse



GEM: 5-moment 2-fluid Maxwell: $\frac{m_i}{m_e} = 1, \tau = 0$, **fine resolution**



GEM: 2-fluid 10-moment ($\frac{m_i}{m_e} = 1$, $\tau = 0.2$)

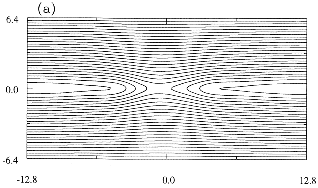


GEM ($\frac{m_i}{m_e} = 25$): kinetic models vs. 5- and 10-moment

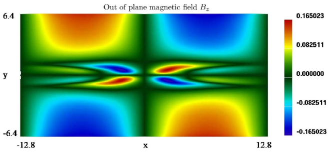
Reconnection rates compare well with with published kinetic results:

model	16% flux reconnected
Vlasov [ScGr06]	$t = 17.7/\Omega_i$:
PIC [Pritchet01]	$t = 15.7/\Omega_i$:
10-moment	$t = 18/\Omega_i$:
5-moment	$t = 13.5/\Omega_i$:

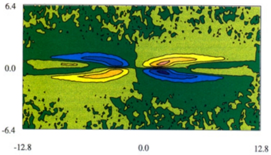
Magnetic field at 16% reconnected



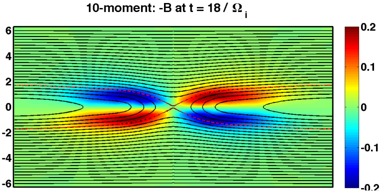
Magnetic field lines for PIC at $\Omega_i t = 15.7$ [Pritchett01]

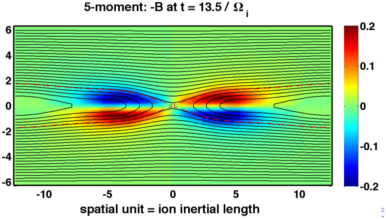


Magnetic field for Vlasov at $\Omega_i t = 17.7$ [ScGr06]

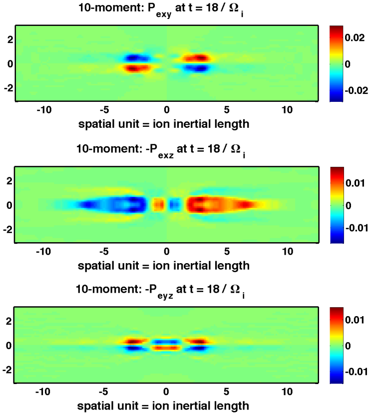


magnetic field of [Pritchett01]

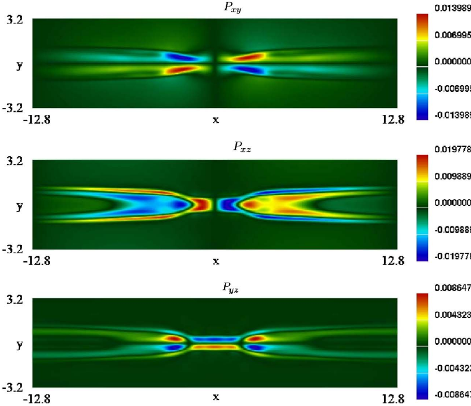




Off-diagonal components of electron pressure tensor

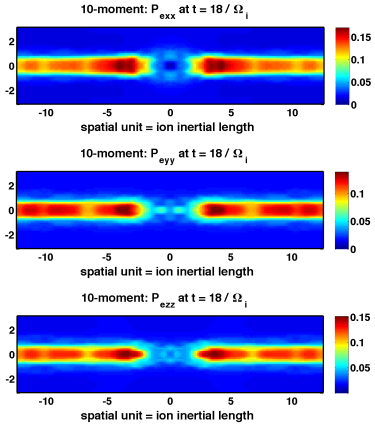


Off-diagonal components of the electron pressure tensor for 10-moment simulation at $\Omega_i t = 18$

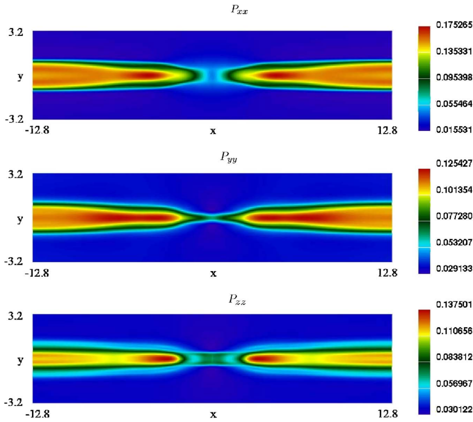


Off-diagonal components of the electron pressure tensor for Vlasov simulation at $\Omega_i t = 17.7$ [ScGr06]

Diagonal components of electron pressure tensor

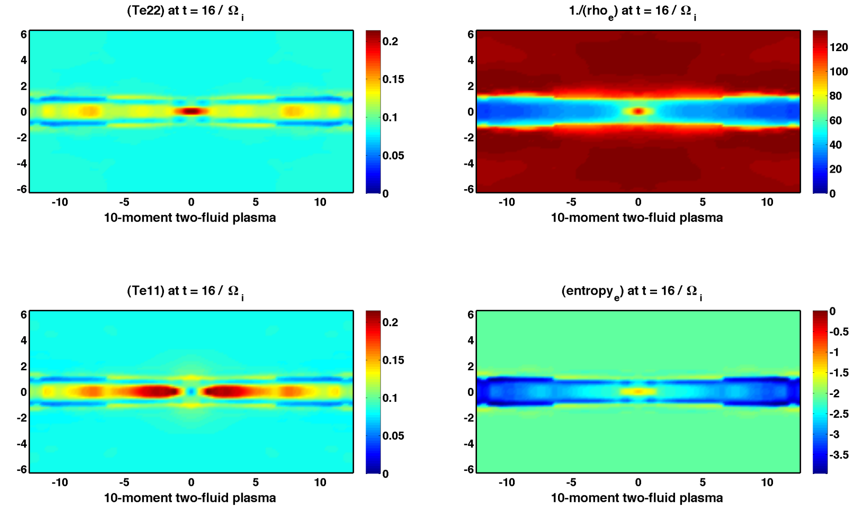


Diagonal components of the electron pressure tensor for 10-moment simulation at $\Omega_i t = 18$

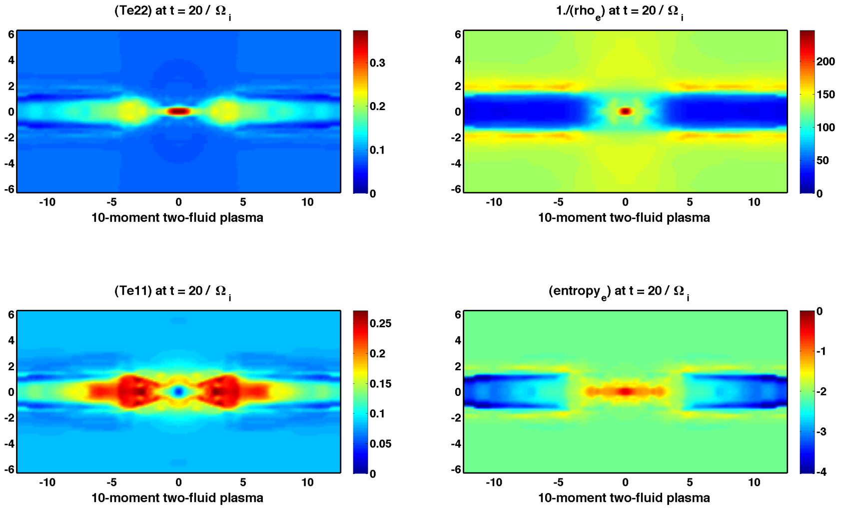


Diagonal components of the electron pressure tensor for Vlasov simulation at $\Omega_i t = 17.7$ [ScGr06]

Problem: the code crashes! Why? Look at electron gas dynamics:

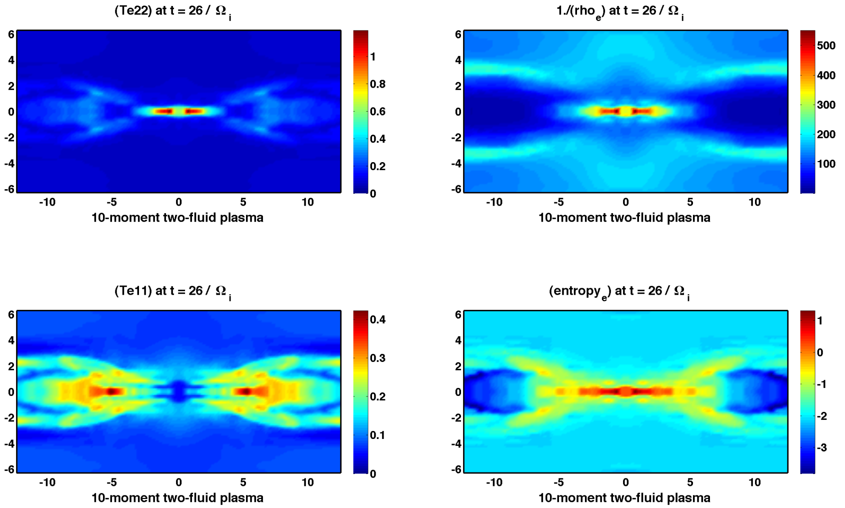


Problem: the code crashes! Why? Look at electron gas dynamics:



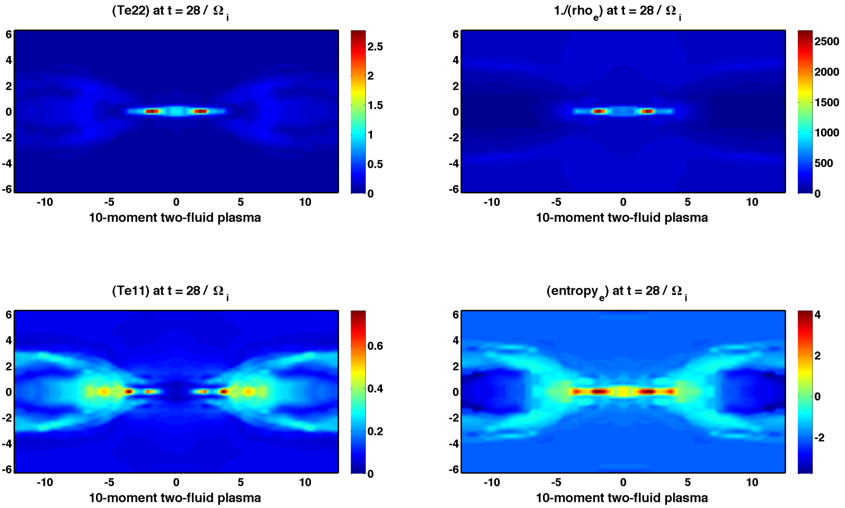
Electron gas at $t = 26$

Problem: the code crashes! Why? Look at electron gas dynamics:



Electron gas at $t = 28$ (just before crashing)

Problem: the code crashes! Why? Look at electron gas dynamics:



Heating singularity ($\frac{m_{\rm i}}{m_{\rm e}} = 25$)

A heating singularity develops between 20% and 50% reconnected flux which crashes the code or produces a central magnetic island.

- $(\mathbb{T}_{\rm e})_{yy}$ becomes large.
- $(\mathbb{T}_{\rm e})_{xx}$ becomes small.
- $\rho_{\rm e}$ becomes small.
- electron entropy becomes large.

These difficulties prompted me to study whether nonsingular steady-state solutions exist for adiabatic plasma models.

Steady magnetic reconnection requires heat flux

Theorem. 2D rotationally symmetric steady magnetic reconnection must be singular in the vicinity of the X-point for an adiabatic model.

Argument. In a steady state solution that is symmetric under 180-degree rotation about the X-point, momentum evolution at the X-point says:

$$\text{rate of reconnection} = \mathbf{E}_3(0) = \frac{-\mathbf{R}_i}{en_i} + \frac{\nabla \cdot \mathbb{P}_i}{en_i}.$$

Assume a nonsingular steady solution. Then at the origin (0) no heat can be produced, so $\mathbf{R}_\mathrm{i} = 0$ at 0. Differentiating entropy evolution twice shows that $\nabla \cdot \mathbb{P}_\mathrm{i} = 0$ at 0. So there is no reconnection.

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