Approaches to multiscale coupling of plasma models

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Abstract: SWIFF proposes a multiscale model to accelerate convergence of macroscale quantities implied by microscale solutions. Time spent computing in transitional regimes at model boundaries should not dominate multiscale simulation. This calls for efficient asymptotic-preserving (AP) microscale schemes. This talk aims to frame and prompt discussion of two essential cases:

1. Making ***kinetic-Maxwell*** *AP with respect to two-fluid-Maxwell* (via relaxation to a Maxwellian distribution of particle velocities), and
2. Making ***two-fluid-Maxwell*** *AP with respect to* ***MHD*** (by making oscillatory time scales and light speed fast.

Outline

1. Models
2. Framework for multiscale methods
3. Kinetic $\rightarrow$ two-fluid
4. two-fluid $\rightarrow$ MHD
5. Conclusion

"Truth": kinetic-Maxwell

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$-c^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$c^{-2}\nabla \cdot \mathbf{E} = \mu_0 \sigma.$$

- **Moments:**

$$\sigma := \sum_s \frac{q_s}{m_s}\rho_s, \qquad \rho_s := \int f_s \, d\mathbf{v},$$
$$\mathbf{J} := \sum_s \frac{q_s}{m_s}\rho_s \mathbf{u}_s, \quad \rho_s \mathbf{u}_s := \int \mathbf{v} f_s \, d\mathbf{v}.$$

- **Kinetic equations:**

$$\partial_t f_s + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \mathbf{a}_s \cdot \nabla_{\mathbf{v}} f_s = \mathcal{C}_s$$

- **Lorentz acceleration:**

$$\mathbf{a}_s = \frac{q_s}{m_s}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

Remarks:

- $\mathcal{C}_s$: "collision operator": incorporates all microscale interactions.
- For two fluids, WLOG $q_i = e$ and $q_e = -e$.
- In my nondimensionalization, proton charge e becomes gyrofrequency and μ_0 becomes plasma beta.
- To change SI to Gaussian units:
  - Choose $\epsilon_0^{-1} = 4\pi$.
  - Replace $\mathbf{B}$ with $\mathbf{B}/c$.
  - So $\mu_0 := \frac{1}{\epsilon_0 c^2} = \frac{4\pi}{c^2}$.
  - For easy reversion to SI, consistently treat π as another name for $(4\epsilon_0)^{-1}$.
- To change Gaussian units to SI:
  - Replace $\mathbf{E}$ with $\sqrt{4\pi\epsilon_0}\mathbf{E}$.
  - Replace $\mathbf{B}$ with $c\sqrt{4\pi\epsilon_0}\mathbf{B}$.
  - Replace q with $\frac{q}{\sqrt{4\pi\epsilon_0}}$.

Modeling parameters

Physical constants that define an ion-electron plasma:

1. e (charge of proton),
2. m_i, m_e (ion and electron mass),
3. c (speed of light),
4. ϵ_0 (vacuum permittivity).

Fundamental parameters that characterize the state of a plasma:

1. n_0 (typical particle density),
2. T_0 (typical temperature),
3. B_0 (typical magnetic field).

Derived quantities:

- $p_0 := n_0 T_0$ (thermal pressure)
- $p_B := \frac{B_0^2}{2\mu_0}$ (magnetic pressure)
- $\rho_s := n_0 m_s$ (typical density).

Collision periods:

- τ_{sp}: expected time for 90-degree deflection of species s via p.

Collisionless time, velocity, and space scale parameters:

plasma frequencies: $\omega_{p,s}^2 := \frac{n_0 e^2}{\epsilon_0 m_s} = \frac{\mu_0 n_0 (ce)^2}{m_s}$,

gyrofrequencies: $\omega_{g,s} := \frac{eB_0}{m_s}$,

thermal velocities: $v_{t,s}^2 := \frac{2p_0}{\rho_s}$,

Alfvén speeds: $v_{B,s}^2 := \frac{2p_B}{\rho_s} = \frac{B_0^2}{\mu_0 m_s n_0}$,

Debye length: $\lambda_D := \frac{v_{t,s}}{\omega_{p,s}} = \sqrt{\frac{\epsilon_0 T_0}{n_0 e^2}}$,

gyroradii: $r_{g,s} := \frac{v_{t,s}}{\omega_{g,s}} = \frac{m_s v_{t,s}}{eB_0}$,

skin depths: $\delta_s := \frac{v_{B,s}}{\omega_{g,s}} = \frac{c}{\omega_{p,s}} = \sqrt{\frac{m_s}{\mu_0 n_s e^2}}$.

plasma $\beta := \frac{p_0}{p_B} = \left(\frac{v_{t,s}}{v_{B,s}}\right)^2 = \left(\frac{r_{g,s}}{\delta_s}\right)^2$.

non-MHD ratio: $\widehat{\lambda} := \frac{v_{B,s}}{c} = \frac{\lambda_D}{r_{g,s}} = \frac{\omega_{g,s}}{\omega_{p,s}} =: \frac{1}{\widehat{\omega}}$.

Nondimensionalization

Choose values for:

m_0	(mass scale)	(e.g. ion mass m_i),
q_0	(charge scale)	(e.g. proton charge e),
B_0	(magnetic field)	(e.g. $\omega_{g,i} m_i / e$),
n_0	(number density)	(e.g. something $\gg 1/x_0^3$),
T_0	(temperature),	(e.g. ion temperature T_i),
τ_0	(relax. period)	(equilibration time scale),
x_0	(space scale)	(anything, e.g. δ_i).

This implies typical values for:

$v_0 = \sqrt{T_0/m_0}$	(velocity scale),
$t_0 = x_0/v_0$	(time scale),
$E_0 = B_0 v_0$	(electric field),
$f_0 := \rho_0 := m_0 n_0$	(mass density),
$C_0 = \rho_0 \tau_0 / t_0$	(collision scale).

Scale parameters are thus:

$$\omega_{p,0}^2 := \frac{n_0 q_0^2}{\epsilon_0 m_0}, \qquad v_t := v_0, \qquad \lambda_D := \frac{v_t}{\omega_{p,0}},$$

$$\omega_g := \frac{q_0 B_0}{m_0}, \qquad v_B^2 := \frac{2p_B}{\rho_0}, \qquad r_g := \frac{v_t}{\omega_g},$$

$$\tau := \frac{\tau_0}{t_0} \qquad \delta_0 := \frac{v_B}{\omega_g}.$$

Making the substitutions

$$\begin{aligned}
t &= \hat{t} t_0, & \mathcal{C}_s &= \widehat{\mathcal{C}} C_0, \\
\mathbf{x} &= \widehat{\mathbf{x}} x_0, & c &= \widehat{c} v_0, \\
q &= \widehat{q} q_0, & \mathbf{v} &= \widehat{\mathbf{v}} v_0, \\
m &= \widehat{m} m_0, & \nabla &= x_0^{-1} \widehat{\nabla} \\
n &= \widehat{n} n_0, & &= x_0^{-1} \nabla_{\widehat{\mathbf{x}}}, \\
\mathbf{B} &= \widehat{\mathbf{B}} B_0, & \nabla_{\mathbf{v}} &= v_0^{-1} \nabla_{\widehat{\mathbf{v}}} \\
\mathbf{E} &= \widehat{\mathbf{E}} B_0 v_0, & &
\end{aligned}$$

in the fundamental equations gives a nondimensional system, where the charge scale e is replaced with gyrofrequency $\widehat{\omega}_g = \frac{e\mathbf{B}_0}{m_0}$ and μ_0 is replaced with plasma beta $\widehat{\mu} = \left(\frac{v_0}{v_B}\right)^2$.

Remark: in nondimensional units, $\omega_g = \frac{q_0 B_0}{m_0}$ simplifies to $\omega_g = e$.

Nondimensionalization in detail

The essence of plasma nondimensionalization is seen in the ***electro-momentum system***, comprised of the two-fluid equations for *momentum* and *electric field evolution*.

Setting $\nabla = 0$ yields

$$d_t \mathbf{u}_s = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) \frac{q_0 B_0 t_0}{m_0}, \quad \frac{q_0 B_0 t_0}{m_0} =: \widehat{\omega}_g$$

$$-\partial_t \mathbf{E} = \left(\sum_s q_s n_s \mathbf{u}_s\right) \frac{t_0 q_0 n_0}{\epsilon_0 B_0}, \quad \frac{t_0 q_0 n_0}{\epsilon_0 B_0} =: \frac{\widehat{\omega}_g}{\widehat{\epsilon}}$$

where we have chosen typical values to nondimensionalize and $E_0 := B_0 u_0$ and $\rho_s = m_s n_s$. Note that

$$\frac{1}{\widehat{\epsilon}} = \underbrace{\left(\frac{n_0 q_0^2}{\epsilon_0 m_0}\right)}_{\omega_p^2} \underbrace{\left(\frac{m_0}{q_0 B_0}\right)}_{\omega_g^{-1}} \frac{t_0}{\widehat{\omega}_g}$$

$$= \left(\frac{\omega_p}{\omega_g}\right)^2 = \widehat{\omega}^2 = \left(\frac{c}{v_B}\right)^2 = \frac{1}{\widehat{\lambda}^2}.$$

Retaining spatial components incorporates wave speeds and completes the picture:

$$-\widehat{c}^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \widehat{\mu}\left(\widehat{\omega}_g \sum_s q_s n_s \mathbf{u}_s\right),$$

$$d_t \mathbf{u}_s + \frac{1}{M_a^2}\frac{\nabla p_s}{\rho_s} = \widehat{\omega}_g \frac{q_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}),$$

where the velocity scale $v_0 := x_0/t_0$ defines the nondimensionalized speed of light $\widehat{c} := c/v_0$, the acoustic quasi-Mach number M_a defined by

$$M_a^2 := \frac{v_0^2 \rho_0}{p_0}$$

can be incorporated into pressure closure, and the magnetic quasi-Mach number M_B defined by

$$\widehat{\mu} := \frac{1}{\widehat{c}^2 \widehat{\epsilon}} = \left(\frac{v_0}{v_B}\right)^2 =: M_B^2$$

can be incorporated into the electromagnetic field.

Plasma beta satisfies $\widehat{\mu} = M_B^2 = \beta M_a^2$. I choose v_0 to be the quasi-acoustic speed $\sqrt{p_0/\rho_0}$, making $M_a = 1$ and $\widehat{\mu} = \beta$.

Model hierarchy

Kinetic-Maxwell

$\Big\downarrow (\widehat{\tau}_{ss} \to 0)$

two-fluid Maxwell

$\Big\downarrow (\widehat{c} \to \infty)$

two-fluid MHD‡

$\Big\downarrow (\widehat{\omega}_g = \widehat{r}_g^{\,-1} \to \infty)$

Ideal MHD‡

Remarks:

- Each model simplification makes some process instantaneous:
  - damping to Maxwellian
  - wave speed
  - oscillations

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‡ these limits are weak (i.e., true after averaging out high-frequency oscillations)

Nondimensionalized **kinetic-Maxwell**

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$-\widehat{c}^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \widehat{\mu}\mathbf{J},$$
$$\widehat{c}^{-2}\nabla \cdot \mathbf{E} = \widehat{\mu}\sigma.$$

- **Moments:**

$$\sigma := \widehat{\omega}_g \sum_s \frac{q_s}{m_s}\rho_s, \qquad \rho_s := \int f_s \, d\mathbf{v},$$
$$\mathbf{J} := \widehat{\omega}_g \sum_s \frac{q_s}{m_s}\rho_s \mathbf{u}_s, \qquad \rho_s \mathbf{u}_s := \int \mathbf{v} f_s \, d\mathbf{v}.$$

- **Kinetic equations:**

$$\partial_t f_s + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \mathbf{a}_s \cdot \nabla_{\mathbf{v}} f_s = \widehat{\tau}^{-1}\mathcal{C}_s$$

- **Lorentz acceleration:**

$$\mathbf{a}_s = \widehat{\omega}_g \frac{q_s}{m_s}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

Nondimensional parameters:

1. **light speed**: $\widehat{c} := \frac{c}{v_0}$,
2. **gyrofrequency** (or gyroradius): $\widehat{\omega}_g := t_0 \omega_{g,0} = \frac{x_0}{r_g} =: \frac{1}{\widehat{r}_g}$
3. **Plasma Beta**: $\widehat{\mu} = \beta = \left(\frac{v_0}{v_B}\right)^2$.
   Or: $\widehat{\epsilon} := \frac{1}{\widehat{\mu}\widehat{c}^2} = \widehat{\lambda}^2$, where $\widehat{\lambda} = \frac{\widehat{\lambda}_D}{\widehat{r}_g} = \frac{\widehat{\omega}_p}{\widehat{\omega}_g} = \frac{c}{v_B}$.
4. **Relaxation period**: $\widehat{\tau} := \frac{\tau_0}{t_0}$.

Remarks:

- Take $\widehat{\mu}$ as constant. (Can eliminate $\widehat{\mu}$ by absorbing $\sqrt{\widehat{\mu}}$ into q_s and $\sqrt{\widehat{\mu}}^{-1}$ into **E** and **B**.)
- MHD is the limit where $\widehat{c} \to \infty$, $\widehat{\omega}_g \to \infty$, and $\widehat{\tau} \to 0$.
- So a kinetic solver can become an MHD solver by artificially dialing these time scales to instantaneous if the solver can efficiently skip over fast time scales.

Now let $\widehat{\tau} \to 0$ to get the two-fluid model.

Nondimensionalized **two-fluid** Euler **Maxwell**

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$-\widehat{c}^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \widehat{\mu}\mathbf{J},$$
$$\widehat{c}^{-2}\nabla \cdot \mathbf{E} = \widehat{\mu}\sigma.$$

- **Moments:**

$$\sigma := \widehat{\omega}_g \sum_s \frac{q_s}{m_s}\rho_s, \qquad \rho_s := \int f_s \, d\mathbf{v},$$
$$\mathbf{J} := \widehat{\omega}_g \sum_s \frac{q_s}{m_s}\rho_s \mathbf{u}_s, \qquad \rho_s \mathbf{u}_s := \int \mathbf{v} f_s \, d\mathbf{v}.$$

- **Euler equations:**

$$\partial_t \rho_s + \nabla \cdot (\rho_s \mathbf{u}_s) = 0,$$
$$\rho_s d_t \mathbf{u}_s + \nabla p_s = \widehat{\omega}_g \frac{q_s}{m_s}\rho_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}),$$
$$d_t^{\mathbf{u}_s} \ln\left(p_s \rho_s^{-\gamma}\right) = 0.$$

Remarks:

- Euler-Maxwell makes the unphysical assumption that:
  - $\widehat{\tau}$ is faster than all other time scales. . .
  - . . . and that τ_{ie} is much slower.
- Non-zero $\widehat{\tau}$ motivates diffusive closures or higher-moment fluid models.
- Non-infinite τ_{ie} motivates resistivity.
- $\widehat{\omega}_g$ may be taken as a proxy for space/time scale or for electron charge.

Now take $\widehat{c} \to \infty$ to get two-fluid MHD (i.e., charge neutrality and vanishing $\partial_t \mathbf{E}$).

Nondimensionalized **2-fluid MHD** ($\widehat{c} \to \infty$)

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u} + \mathbf{E}') = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\mathbf{J} = \widehat{\mu}^{-1} \nabla \times \mathbf{B}$$

Ohm's law

$$\begin{aligned}\mathbf{E}' &= \frac{m_\mathrm{i} - m_\mathrm{e}}{(e\widehat{\omega}_g)\rho} \mathbf{J} \times \mathbf{B} \\ &+ \frac{1}{(e\widehat{\omega}_g)\rho} \nabla (m_\mathrm{e} p_\mathrm{i} - m_\mathrm{i} p_\mathrm{e}) \\ &+ \frac{m_\mathrm{i} m_\mathrm{e}}{(e\widehat{\omega}_g)^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_\mathrm{i} - m_\mathrm{e}}{(e\widehat{\omega}_g)\rho} \mathbf{JJ} \right) \right]\end{aligned}$$

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0,$$
$$\rho d_t \mathbf{u} + \nabla (p_\mathrm{i} + p_\mathrm{e}) + \nabla \cdot \mathbb{P}^\mathrm{d} = \mathbf{J} \times \mathbf{B}.$$

Pressure for each species:

$$d_t^{\mathbf{u}_\mathrm{s}} \ln \left(p_\mathrm{s} \rho_\mathrm{s}^{-\gamma} \right) = 0.$$

Implied total pressure evolution:

$$d_t \ln \left(p \rho^{-\gamma} \right) + p^{-1} \nabla \cdot \mathbf{q}^\mathrm{d} = 0.$$

Other Definitions:

$$\begin{aligned} d_t &:= \partial_t + \mathbf{u} \cdot \nabla, \\ d_t^{\mathbf{u}_\mathrm{s}} &:= \partial_t + \mathbf{u}_\mathrm{s} \cdot \nabla, \\ \mathbb{P}^\mathrm{d} &:= \textstyle\sum_\mathrm{s} \rho_\mathrm{s} \mathbf{w}_\mathrm{s} \mathbf{w}_\mathrm{s} \approx m_\mathrm{red} n \mathbf{ww} \\ \mathbf{w} &:= \mathbf{u}_\mathrm{i} - \mathbf{u}_\mathrm{e} \approx \frac{\mathbf{J}}{(e\widehat{\omega}_g) n} \end{aligned}$$

(by quasineutrality),

$$\begin{aligned} m_\mathrm{red}^{-1} &:= m_\mathrm{e}^{-1} + m_\mathrm{i}^{-1}. \\ \mathbf{q}^\mathrm{d} &:= \textstyle\sum_\mathrm{s} \left(\mathbf{w}_\mathrm{s} (\mathcal{E}_\mathrm{s} + p_\mathrm{s}) \right) \end{aligned}$$

Now take $\widehat{\omega}_g \to \infty$ to get Ideal MHD (i.e., ideal Ohm's law and vanishing $\mathbf{q}^\mathrm{d}$ and $\mathbb{P}^\mathrm{d}$).

Equations of **Ideal MHD** ($\widehat{\omega}_g \to \infty$)

Electromagnetism

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0,$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\mathbf{J} = \widehat{\mu}^{-1} \nabla \times \mathbf{B}$$

mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$
$$\rho d_t \mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B}$$

Pressure

$$d_t \ln \left(p\rho^{-\gamma}\right) = 0.$$

Energy:

$$\partial_t \mathcal{E} + \nabla \cdot (\mathbf{u}(\mathcal{E} + p)) = \mathbf{J} \cdot \mathbf{E}$$

Remarks:

- Ideal MHD is a weak limit
- Ideal MHD is hyperbolic and can develop shocks, in which case full energy evolution and conservation form should be used.

Outline

Coarse representation

- Philosophy: only resolve what is needed
- Coarse representations are justified
  - when microscale processes are “along for the ride” and
  - when the effect of micro on macro can be inferred from macro quanitites.
- Kinds of coarsening:
  - discretization (mesh, time step, order of accuracy), and
  - physics (model):
    - representation (of particle density) and
    - closure (model used to evolve the representation).
- Model coarsening is justified by fast processes:
  - (small reduction): *fast plasma frequency* (neutrality),
  - (some reduction): *fast gyrofrequency* (gyrokinetic reduction), and
  - (great reduction): *fast collisions* (kinetic $\rightarrow$ fluid).

Multiscale convergence

What does it mean for a multiscale algorithm to converge?

- The quantities that we care about are defined in terms of the coarsest representation (e.g. MHD fluid moments).
- Goal: accurately predict the projection of the fine solution onto the coarse model.
- Multiscale algorithm: uses the coarse model to accelerate convergence of the coarse projection of the fine solution.

An explicit algorithm must resolve the fastest process in the system. This suggests two approaches:

1. Explicit algorithm approach:
    - Artificially *slow down* fast processes.
    - Justification: if you don't need to resolve it then why not slow it down?
    - Issue: trying to stitch to a model that makes the process instantaneous:
        - conservation is sacrificed to maintain regularity or positivity.
        - thick sponge region needed to absorb impedance mismatch.
        - feasible only for coarse resolution.
    - Artificially slowing down processes is appropriate where sticking to a single model.
2. Asymptotic-preserving (AP) implicit algorithm approach:
    - Use an implicit method to step over fast processes.
    - *speed up* fast processes so that the micro solver becomes a macro solver.
    - Advantages:
        - can greatly reduce or even eliminate the sponge region.
        - can maintain conserved quantities.
    - Issues:
        - more complicated to implement
        - convergence becomes slow in transitional regimes

Implicit schemes: full versus semi-implicit

How do you deal with slow convergence in transitional regimes?

Semi-implicit approach.

- separate out and linearize fast processes via operator splitting
- philosophy: fast processes are slaved precisely because they are approximately linear.
- advantages:
  - non-iterative
  - simple to implement
- issues:
  - time-splitting error:
    - ostensibly not a problem (not trying to resolve the split-off processes anyway) –
    - but near-equilibrium states need a well-balanced scheme.
  - divergence constraints are not exactly maintained.

Fully implicit approach.

- Allows conforming discretization.
- Use semi-implicit or macroscale solution as predictor.
- Use algebraic multigrid to accelerate convergence of the microscale model. See [BraLi11].
- Want a bound on number of iterations needed for convergence that is independent of the choice of fast time scale.

For efficient micro-macro coupling, the microscale discretization needs to be **asymptotic-preserving (AP)** with respect to the macroscale model:

$$\begin{array}{ccc} \text{micro-scheme}(\epsilon, h) & \xrightarrow{h\to 0} & \text{micro-physics}(\epsilon) \\ \Big\downarrow{\scriptstyle \epsilon\to 0} & & \Big\downarrow{\scriptstyle \epsilon\to 0} \\ \textbf{macro-scheme}(h) & \xrightarrow{h\to 0} & \text{macro-physics} \end{array}$$

The micro-scheme is ***efficiently AP*** with respect to the macro-physics if it converges to a limiting macro-scheme that is a consistent, stable, efficient solver for the macro-physics model.

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Kinetic → two-fluid

- Making kinetic models AP with respect to two-fluid models is “easy”:
  - Relax to the assumed distribution at a tunable rate.
  - Relax to Maxwellian (or Kappa distribution) to get two-fluid Euler.
- Challenge: kinetic solver should become an *efficient* fluid solver in the fluid limit.
  - cost of limit solver needs to be on same order as cost of fluid solver *for efficient convergence*.
  - so kinetic representation must be able to efficiently represent assumed velocity spread of fluid model.
  - natural way: δf method. (See [CreCroLem12] for a successful Vlasov-Poisson implementation.)
  - alternative: periodically resample using particles with e.g. Gaussian shape in phase space. (See [Hewett02].) Periodic resampling based on linear deformation of particle shape turns a second-order PIC code into a second-order Vlasov solver without modification of the PIC algorithm used to evolve particles and fields. (See [Pinto12], [PintoSonFriGroLun12].)
- Side-benefits:
  - less noise
  - fewer particles needed
- Challenges:
  - δf: definition of projection operators to exchange information between fluid and kinetic components
  - resampling: need efficient resampling of particles. Can use wavelet decomposition. Use particle splitting to delay global resampling.

Outline

Two-fluid splitting strategy

Two-fluid Euler Maxwell looks like

$$\partial_t q + \nabla \cdot \mathbf{f} = s.$$

Operator splitting alternately solves the hyperbolic part,

$$(H)\quad \partial_t q + \nabla \cdot \mathbf{f} = 0$$

and the source term part,

$$(S)\quad \partial_t q = s.$$

Source term:

- linear ODE with constant coefficients
- imaginary eigenvalues
- can step over plasma period and gyroperiods without iteration.

Hyperbolic part:

- homogeneous Maxwell is linear,
- so can step over light speed without iteration.

Advantage: a noniterative method can step over all time scales needed for MHD limit.

Issues:

- Divergence constraints are ignored. (Evolve correction potentials.)
- Time splitting must be well-balanced.
- Trapezoid rule looks promising.

Simulation results (Harish Kumar)

1. Brio-Wu simulations using an explicit method. (Slides 27–34 posted at `http://www.danlj.org/eaj/math/research/presentations/hyp2008/talk.pdf`.)
2. Harish Kumar's simulations using time splitting. (Slides presented to MATH-CCS at RWTH Aachen on 30th October, 2012.)
3. Kumar and Mishra IMEX simulations. (Figure 5 on page 21 of [KumarMishra11].)

Outline

Summary of proposed IMEX scheme

Non-iterative two-fluid $\stackrel{\text{AP}}{\to}$ MHD:

- split off source term and solve exactly or via trapezoid rule.
- solve homogeneous Maxwell via non-iterative implicit solver.

Non-iterative Kinetic $\stackrel{\text{AP}}{\to}$ two-fluid:

- Embed fluid model and relax δf to zero to make kinetic AP with respect to fluid
- Split off collision operator
- δf PIC plus operator splitting

Further possibilities for kinetic IMEX solver:

- Use a higher-moment model for embedded fluid model.
- Use periodic resampling to turn PIC code into second-order Vlasov solver.
- Apply operator splitting to PIC to get kinetic $\stackrel{\text{AP}}{\to}$ MHD directly.

References

The following article describes an IMEX scheme for two-fluid-Maxwell that gives efficient and close agreement with MHD for the Brio-Wu 1D shock problem.

[KumarMishra11] Harish Kumar and Siddartha Mishra, *Entropy stable numerical schemes for two-fluid plasma equations,* J Sci Comput (2012), DOI 10.1007/s10915-011-9554-7

The following article describes an asymptotic-preserving δf-type kinetic solver that becomes an efficient fluid solver in the fluid limit.

[CreCroLem12] Anais Crestetto, Nicolas Crouseilles, and Mohammed Lemou, *Kinetic/fluid micro/macro numerical schemes for Vlasov-Poisson-BGK equation using particles*, submitted to AIMS' Journals, version 1, Sep 2012.

The following guide explains how to accelerate convergence of implicit methods with multigrid.

[BraLi11] Achi Brandt and Oren Livne, *Multigrid Techniques: 1984 Guide with Applications to Fluid Dynamics, Revised Edition*, SIAM, 2011. Available online at `http://www.ec-securehost.com/SIAM/CL67.html`

The following articles explain how to turn a PIC solver into a second-order-accurate Vlasov solver without modifying the particle evolution algorithm.

[Pinto12] Martin Campos Pinto, *Smooth particle methods without smoothing*, arXiv:1112.1859v2, 9 Mar 2012.

[PintoSonFriGroLun12] M. Campos Pinto, E. Sonnendrücker, A. Friedman, D. Grote, and S. Lund, *Noiseless Vlasov-Poisson simulations with linearly transformed particles*, arXiv1211.5047v1, 21 Nov 2012.

This article describes use of particles with Gaussian shape in phase space:

[Hewett02] Dennis W. Hewett, *Fragmentation, merging, and internal dynamics for PIC simulation with finite size particles*, Journal of Computational Physics 189 (2003) 390–426.