A Fast Shock-Capturing Algorithm for a Two-Fluid Plasma Model

E. Alec Johnson
Department of Mathematics, UW-Madison
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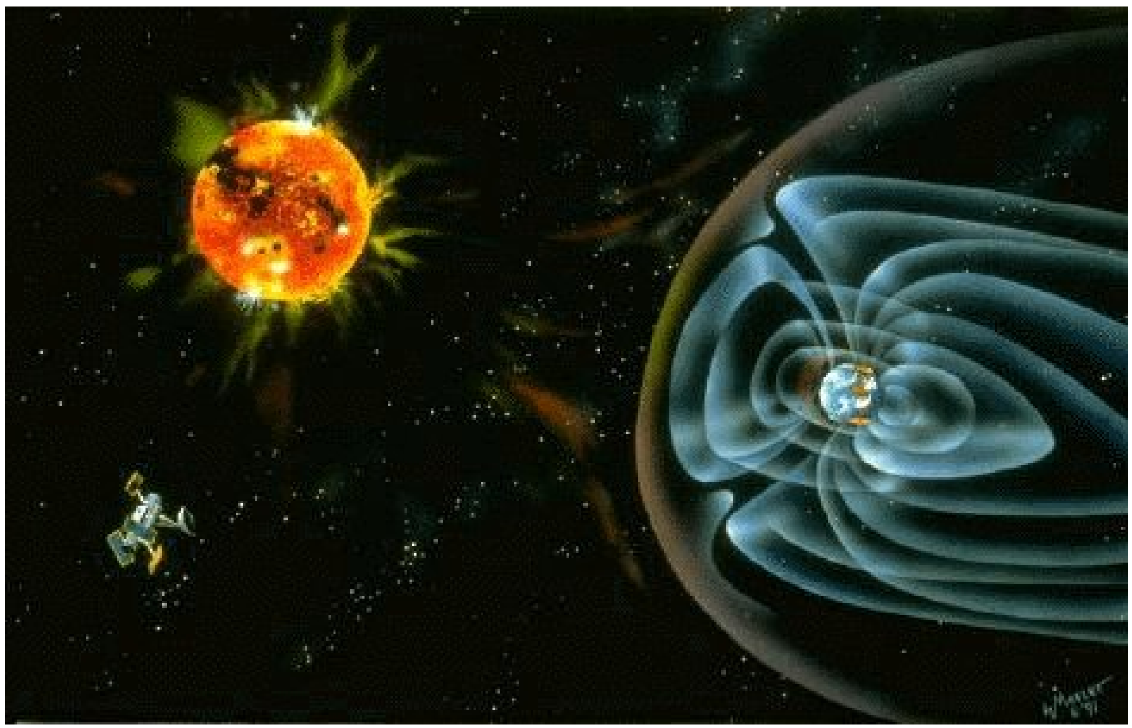


Outline

① Problem: space weather and fast reconnection

② Physical model: two-fluid plasma.

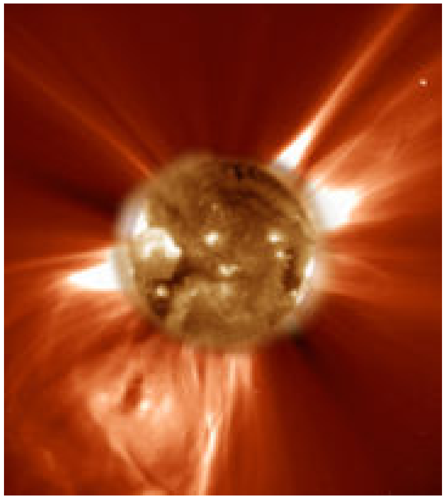
③ Computations: Riemann problem.

④ Future work: fast GEM solver.

Problem: Space weather

Broad goal: to model **space weather**.

- Earth bombarded with **solar wind**.
  1. *charged*: mostly protons or electrons.
  2. *sparse*: 5-10 protons (or electrons) per cm^3.
  3. *fast-moving*: proton velocities of 200–800 km/s.
- Solar wind varies dramatically.
- Solar storms cause **geomagnetic storms**.



Coronal Mass Ejection.

Problem: Space weather: Human effects.

How does a geomagnetic storm affect us?

- Produces polar auroras.
- Deflects compass readings.
- Generates large ground currents, affecting geologic exploration.
- Damages transformers in power grids. (HydroQuébec, 1989)
- Disrupts satellite communications and navigation systems, including GPS.
- Threatens spacecraft and astronauts.

Problem: Space plasma.

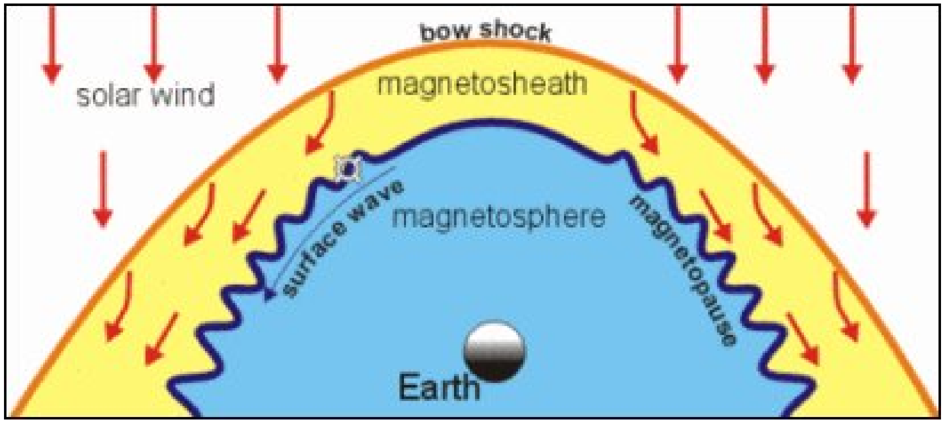
Solar wind is a form of plasma.

- **Plasma** is gas with enough charged particles so that electromagnetism affects its motion.
- Space plasmas develop magnetic fields. Charged particles spiral around magnetic field lines. So **particles are effectively constrained to move along magnetic field lines**, and magnetic field lines are bound to the plasma.
- Space plasmas are approximately **collisionless**: particles interact mostly with the overall electromagnetic field rather than by collisions with individual particles.

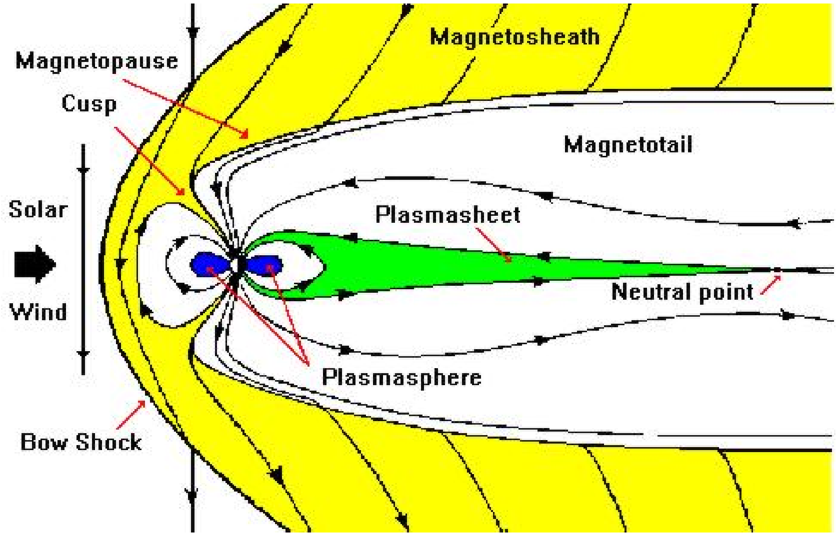
Problem: Earth's magnetosphere

How does the solar wind interact with Earth's magnetic field?

- The **magnetosphere** is the region around Earth whose magnetic field lines pass through Earth. Outside this region is the **Interplanetary Magnetic Field (IMF)**.
- The **magnetopause** is the boundary between the magnetosphere and the IMF.
- **Since space plasma is tightly constrained to flow along magnetic field lines, Earth's magnetosphere largely shields us from the solar wind** (and is distorted in the process).
- Solar wind is decelerated at the bow shock.
- Solar wind is generally deflected around the magnetopause.
- But reconnection of magnetic field lines allows plasma to cross the magnetopause into Earth's magnetosphere.

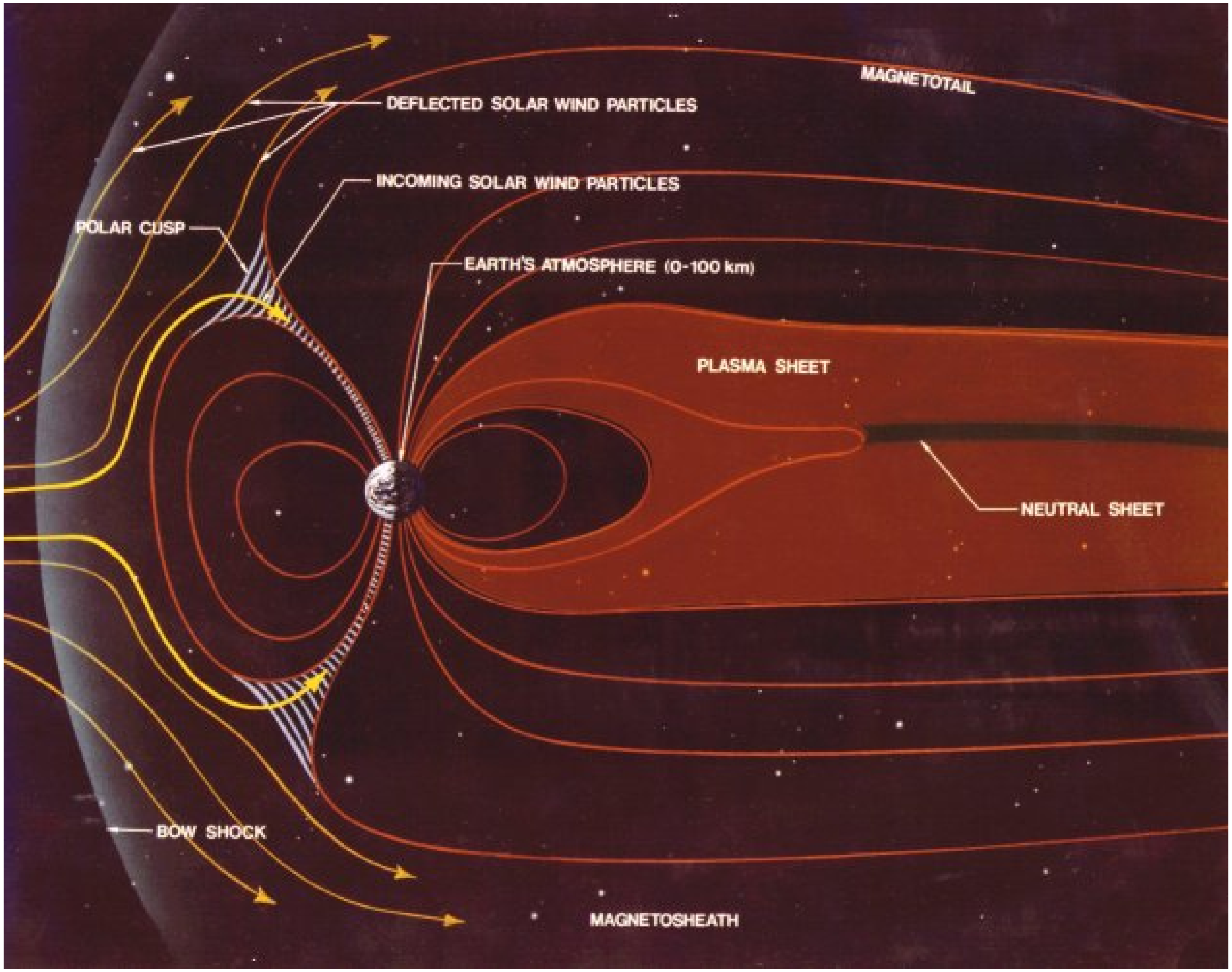


(cross-section along ecliptic)

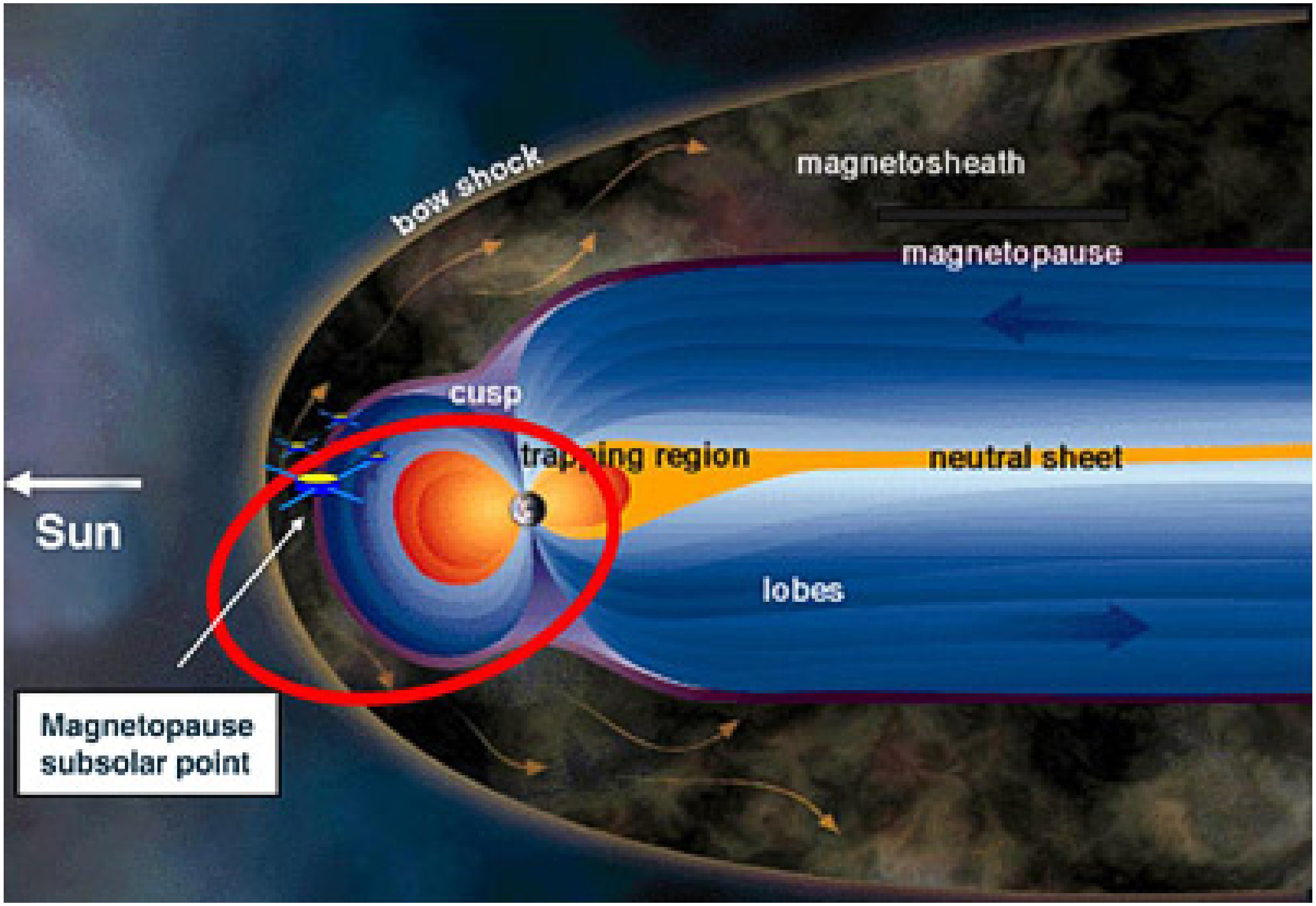


(cross-section along polar axis)

Problem: Space weather: Magnetosphere.



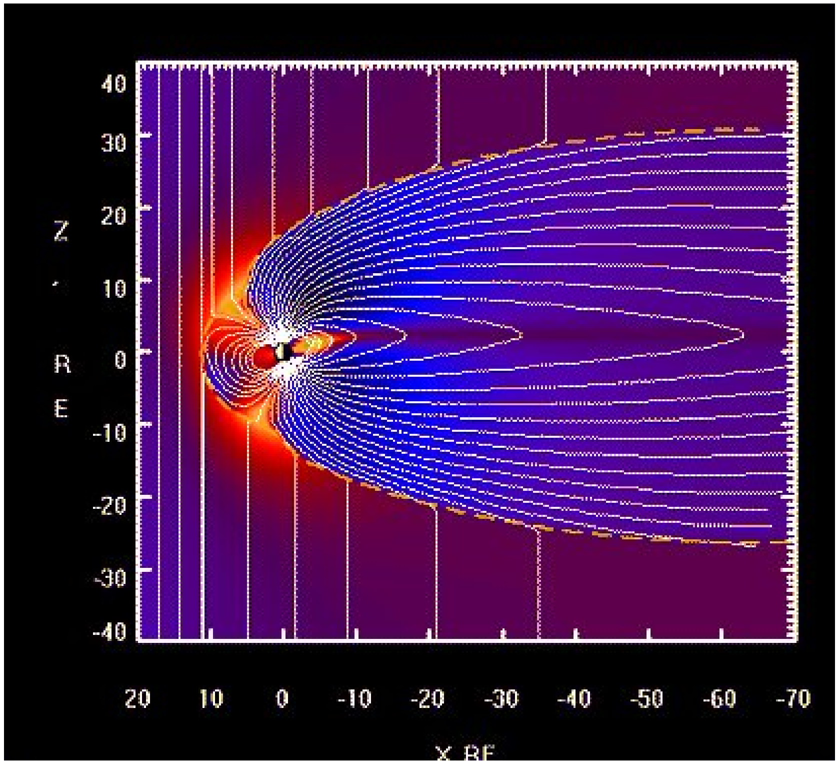
Problem: Space weather: Magnetosphere.



The red circle indicates the position of the four satellites of the European Space Agency's Cluster fleet.

Credits: ESA

Problem: Magnetic reconnection.



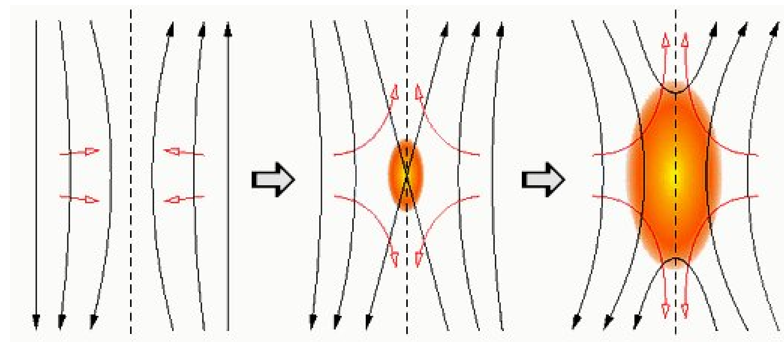
Solar wind reconnecting at magnetopause

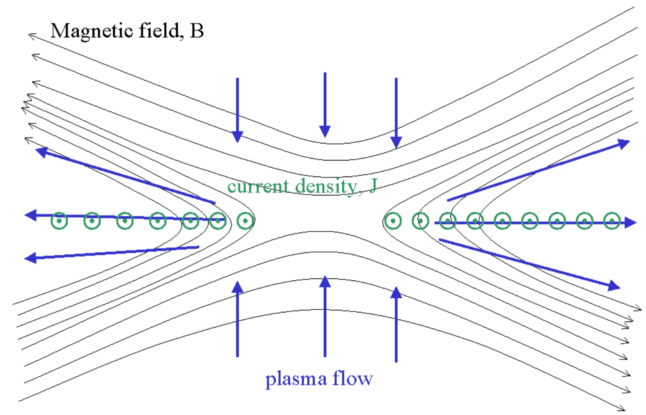
14-Aug-2007 21:03:37 UT
Schematic of Reconnection
Date: 03 Feb 2005
Satellite: Cluster
Depicts: Reconnecting field lines

Problem: Magnetic reconnection.

What is magnetic reconnection?

- Magnetic reconnection is reconfiguration of the topology of magnetic field lines.
- How do field lines reconnect? (2D scenario).
  1. Adjacent antiparallel lines approach.
  2. The lines form an X and cancel, breaking them in two.
  3. The adjacent broken halves join.
  4. These new sharply angled lines form a "slingshot"; the region of canceled field propagates rapidly.
  5. This rapid cancellation of magnetic field explosively releases the energy stored in the magnetic field.





Problem: Fast magnetic reconnection.

Specific goal: to develop a fast, shock-capturing algorithm that resolves **fast magnetic reconnection**.

① Solar wind produces strong shocks.

② Fast magnetic reconnection is critical to modeling space weather events.

  (a) Fast reconnection seems to make violent solar storms possible.

  (b) Reconnection is the primary mechanism that allows gusts of solar wind to penetrate the magnetosphere and generate geomagnetic storms.

③ Models that resolve fast reconnection (and by implication fast waves) tend to be computationally expensive due to the need for a short time step.

④ Reconnection typically is restricted to isolated regions of space, and elsewhere cheaper models that don't resolve fast waves are accurate enough.

⑤ Our challenge: **Can we selectively resolve fast waves only in regions where magnetic reconnection is occurring, and elsewhere use a coarser time step?**

Plasma models

We seek the simplest, most computationally inexpensive model that exhibits fast reconnection.

What models are available?

① **Two-fluid** models regard the plasma as an electron fluid and an ion fluid which occupy the same space.

② The **collisionless two-fluid** model assumes that the two fluids pass through one another freely and only affect one another indirectly by means of their interaction with the electromagnetic field.

③ **One-fluid** models essentially regard the plasma as a charge-neutral fluid which conducts electricity.

Choice of plasma model.

Why did we choose the collisionless two-fluid model?

Plasma models: accuracy versus expense.

① **Collisionless Two-fluid** model. (Admits fast reconnection.)
- Most accurate.
- Expensive because of light waves.

② **One-fluid** models, i.e.
Magnetohydrodynamics (MHD).
(a) **Hall MHD.** (Admits fast reconnection.)
(b) **Resistive MHD.** (Reconnection converges to the correct steady state, but too slowly by orders of magnitude.)
(c) **Ideal MHD.** (Does not admit reconnection.)

We chose to focus on the **collisionless two-fluid model** rather than Hall MHD because of its simplicity.

We compare our computations with ideal MHD, which should be sufficiently accurate in the majority of regions where reconnection and strong shocks are absent.

Model equations: Conservation law framework

We express our equations as conservation (balance) laws.

$$\underline{q}_t + \underline{\nabla} \cdot \underline{\underline{f}}(\underline{q}) = \underline{s}(\underline{q}).$$

- $\underline{q}$ is the state, i.e., a tuple of conserved "stuff",
- $\underline{\underline{f}}$ is the flux, i.e., the rate at which each type of stuff flows, and
- $\underline{s}$ is the source term, i.e., the rate at which each kind of stuff is produced or destroyed.

If we partition space into discrete cells, partition time into discrete time intervals, and integrate this equation over each cell C and time interval $[t_n, t_{n+1}]$, this equation says that the change in the amount of stuff in a cell over one time step equals the net amount that flowed into the cell across the cell interfaces plus the net amount of stuff that was produced inside the cell.

$$\int_C \underline{q}(t_{n+1}) = \int_C \underline{q}(t_n) - \int_C \mathbf{n} \cdot \underline{\underline{f}} + \int_C \underline{s}$$

Model equations: Two-fluid model

The two-fluid model consists of gas dynamics for each of the two fluids, coupled to Maxwell's equations by means of source terms consisting of the Lorentz force, the charge density, and the current and displacement currents. The gas dynamics equations are

$$\underbrace{\partial_t \begin{bmatrix} \rho_s \\ \rho_s \mathbf{v}_s \\ \mathcal{E}_s \end{bmatrix}}_{\text{conserved}} + \underbrace{\nabla \cdot \begin{bmatrix} \rho_s \mathbf{v}_s \\ \rho_s \mathbf{v}_s \mathbf{v}_s + p_s \mathbb{I} \\ \mathbf{v}_s (\mathcal{E}_s + p_s) \end{bmatrix}}_{\text{hyperbolic flux}} = \underbrace{\begin{bmatrix} 0 \\ \frac{q_s}{m_s} \rho_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \\ \frac{q_s}{m_s} \rho_s \mathbf{v}_s \cdot \mathbf{E} \end{bmatrix}}_{\text{electromagnetic source}},$$

where $s = i$ (ion) or e (electron), $\frac{q_s}{m_s}$ is charge-to-mass ratio, ρ is mass density, $\mathbf{v}$ is fluid velocity, $\mathcal{E}$ is energy, p is pressure, and $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic field, We assume the ideal gas constitutive relations $\mathcal{E}_s = \frac{p_s}{\gamma_s - 1} + \frac{1}{2} \rho_s v_s^2$. The charge density and the current density of each species are given by the relations $\sigma_s = \frac{q_s}{m_s} \rho_s$ and $\mathbf{J}_s = \frac{q_s}{m_s} \rho_s \mathbf{v}_s$.

Maxwell's equations for the evolution of the electromagnetic field are

$$\underbrace{\partial_t \begin{bmatrix} c\mathbf{B} \\ \mathbf{E} \end{bmatrix} + c\nabla \times \begin{bmatrix} \mathbf{E} \\ -c\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathbf{J}/\epsilon_0 \end{bmatrix}}_{\text{evolution equations}} \text{ and } \underbrace{\nabla \cdot \begin{bmatrix} c\mathbf{B} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma/\epsilon_0 \end{bmatrix}}_{\text{constraint equations}},$$

where $\mathbf{B}$ = magnetic field, $\mathbf{E}$ = electric field, $\sigma = \sigma_i + \sigma_e$ = net charge density, $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_e$ = net current, c = light speed, and ϵ_0 = permittivity of free space.

Model equations: Nondimensionalized two-fluid model

We nondimensionalized the equations by choosing typical values for an ion. We get:

$$\partial_{\hat{t}} \begin{bmatrix} \hat{\rho}_i \\ \hat{\rho}_e \\ \hat{\rho}_i \hat{\mathbf{v}}_i \\ \hat{\rho}_e \hat{\mathbf{v}}_e \\ \hat{\mathcal{E}}_i \\ \hat{\mathcal{E}}_e \end{bmatrix} + \widehat{\nabla\cdot} \begin{bmatrix} \hat{\rho}_i \hat{\mathbf{v}}_i \\ \hat{\rho}_e \hat{\mathbf{v}}_e \\ \hat{\rho}_i \hat{\mathbf{v}}_i \hat{\mathbf{v}}_i + \hat{p}_i \mathbb{I} \\ \hat{\rho}_e \hat{\mathbf{v}}_e \hat{\mathbf{v}}_e + \hat{p}_e \mathbb{I} \\ \hat{\mathbf{v}}_i \left(\hat{\mathcal{E}}_i + \hat{p}_i\right) \\ \hat{\mathbf{v}}_e \left(\hat{\mathcal{E}}_e + \hat{p}_e\right) \end{bmatrix} = \frac{1}{\hat{r}_L} \begin{bmatrix} 0 \\ 0 \\ \hat{\rho}_i (\hat{\mathbf{E}} + \hat{\mathbf{v}}_i \times \hat{\mathbf{B}}) \\ -\frac{m_i}{m_e} \hat{\rho}_e (\hat{\mathbf{E}} + \hat{\mathbf{v}}_e \times \hat{\mathbf{B}}) \\ \hat{\rho}_i \hat{\mathbf{v}}_i \cdot \hat{\mathbf{E}} \\ -\frac{m_i}{m_e} \hat{\rho}_e \hat{\mathbf{v}}_e \cdot \hat{\mathbf{E}} \end{bmatrix},$$

$$\partial_{\hat{t}} \begin{bmatrix} \hat{c}\hat{\mathbf{B}} \\ \hat{\mathbf{E}} \end{bmatrix} + \hat{c}\widehat{\nabla\times} \begin{bmatrix} \hat{\mathbf{E}} \\ -\hat{c}\hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\hat{\mathbf{J}}/\epsilon \end{bmatrix}, \text{ and } \hat{\nabla} \cdot \begin{bmatrix} \hat{c}\hat{\mathbf{B}} \\ \hat{\mathbf{E}} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{\sigma}/\epsilon \end{bmatrix}.$$

Here $r_L := \frac{m_0 v_0}{q_0 B_0}$ is the **Larmor radius**, the radius of curvature of the circular motion of a typical ion moving perpendicular to a characteristic magnetic field, $\epsilon := \hat{r}_L \widehat{\lambda_D}^2$ plays the role of permittivity, and $\widehat{\lambda_D}$ is the ratio of the Debye length to the Larmor radius.

The **Debye length**, $\sqrt{\frac{\epsilon_0 m_0 v_0^2}{n_0 q_0^2}}$, is the distance scale over which electrons screen out electric fields in plasmas (i.e. the distance scale over which significant charge separation can occur). n_0 is a typical value for number density.

Model equations: Ideal MHD

If we consider an asymptotic limit of the two-fluid equations as the Larmor radius r_L goes to zero, we obtain the following assumptions used in deriving the MHD equations:

① $\sigma \approx 0$ (quasineutrality)

② $\partial_t \mathbf{E} \approx 0$ (Ampere's law)

③ $\mathbf{E} \approx \mathbf{B} \times \mathbf{v}$ (Ohm's law)

④ $E^2 \approx 0$ (small $\mathbf{E}$)

The work we report compares the two-fluid plasma model with ideal MHD as we take the Larmor radius r_L to zero. The full system of ideal MHD equations is

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\mathbf{v} \\ \tilde{\mathcal{E}} \\ \mathbf{B}\end{bmatrix} + \nabla \cdot \underbrace{\begin{bmatrix}\rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v} + \tilde{p}\,\mathbb{I} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B} \\ \mathbf{v}\left(\tilde{\mathcal{E}} + \tilde{p}\right) - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}\cdot\mathbf{v} \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}\end{bmatrix}}_{\text{hyperbolic flux}} = 0$$

and the physical constraint $\nabla \cdot \mathbf{B} = 0$, where ρ is the mass density, $\mathbf{v}$ is the fluid velocity field, $\tilde{\mathcal{E}} := \mathcal{E} + \frac{1}{2\mu_0}B^2$ is the total energy (gas-dynamic energy plus magnetic energy), $\mathbf{B}$ is the magnetic field, and $\tilde{p} := p + \frac{1}{2\mu_0}B^2$ is the total pressure (gas-dynamic pressure plus magnetic pressure). The gas-dynamic pressure is $p = (\gamma - 1)(\mathcal{E} - \frac{1}{2}\rho v^2)$, where $\gamma = \frac{5}{3}$ is the ratio of specific heats.

Numerical Method: Finite Volume.

We used an explicit, finite-volume, shock-capturing numerical method.

- Implemented in Randall LeVeque's CLAWPACK (Conservation LAW PACKage).
- Second-order accurate for smooth data, first-order accurate for shocks.
- For 2-fluid, used Strang time-splitting to handle source term.

Numerical Method: Choice of problem.

We computed solutions to a 1-dimensional Riemann problem.

For MHD the initial conditions to the left and right of zero were:

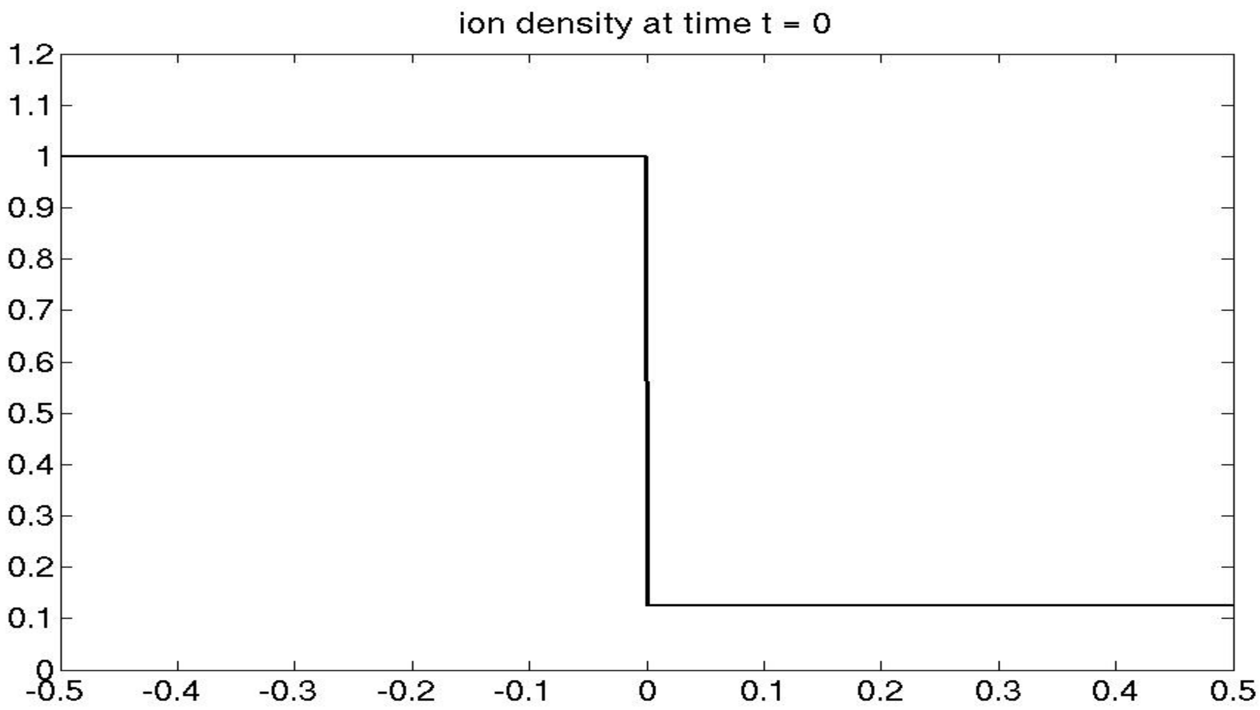
$$\begin{bmatrix}\rho\\ v^1\\ v^2\\ v^3\\ p\\ B^1\\ B^2\\ B^3\end{bmatrix}_{\text{left}} = \begin{bmatrix}1.0\\ 0\\ 0\\ 0\\ 1.0\\ 0.75\\ 1.0\\ 0\end{bmatrix} \text{ and } \begin{bmatrix}\rho\\ v^1\\ v^2\\ v^3\\ p\\ B^1\\ B^2\\ B^3\end{bmatrix}_{\text{right}} = \begin{bmatrix}0.125\\ 0\\ 0\\ 0\\ 0.1\\ 0.75\\ -1.0\\ 0\end{bmatrix}$$

The equivalent initial conditions for the two-fluid model are:

$$\begin{bmatrix}\rho_i\\ v_i^1\\ v_i^2\\ v_i^3\\ p_i\\ \rho_e\\ v_e^1\\ v_e^2\\ v_e^3\\ p_e\\ B^1\\ B^2\\ B^3\\ E^1\\ E^2\\ E^3\end{bmatrix}_{\text{left}} = \begin{bmatrix}1.0\\ 0\\ 0\\ 0\\ 0.5\\ 1.0\frac{m_e}{m_i}\\ 0\\ 0\\ 0\\ 0.5\\ 0.75\\ 1.0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \text{ and } \begin{bmatrix}\rho_i\\ v_i^1\\ v_i^2\\ v_i^3\\ p_i\\ \rho_e\\ v_e^1\\ v_e^2\\ v_e^3\\ p_e\\ B^1\\ B^2\\ B^3\\ E^1\\ E^2\\ E^3\end{bmatrix}_{\text{right}} = \begin{bmatrix}0.125\\ 0\\ 0\\ 0\\ 0.05\\ 0.125\frac{m_e}{m_i}\\ 0\\ 0\\ 0\\ 0.05\\ 0.75\\ -1.0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

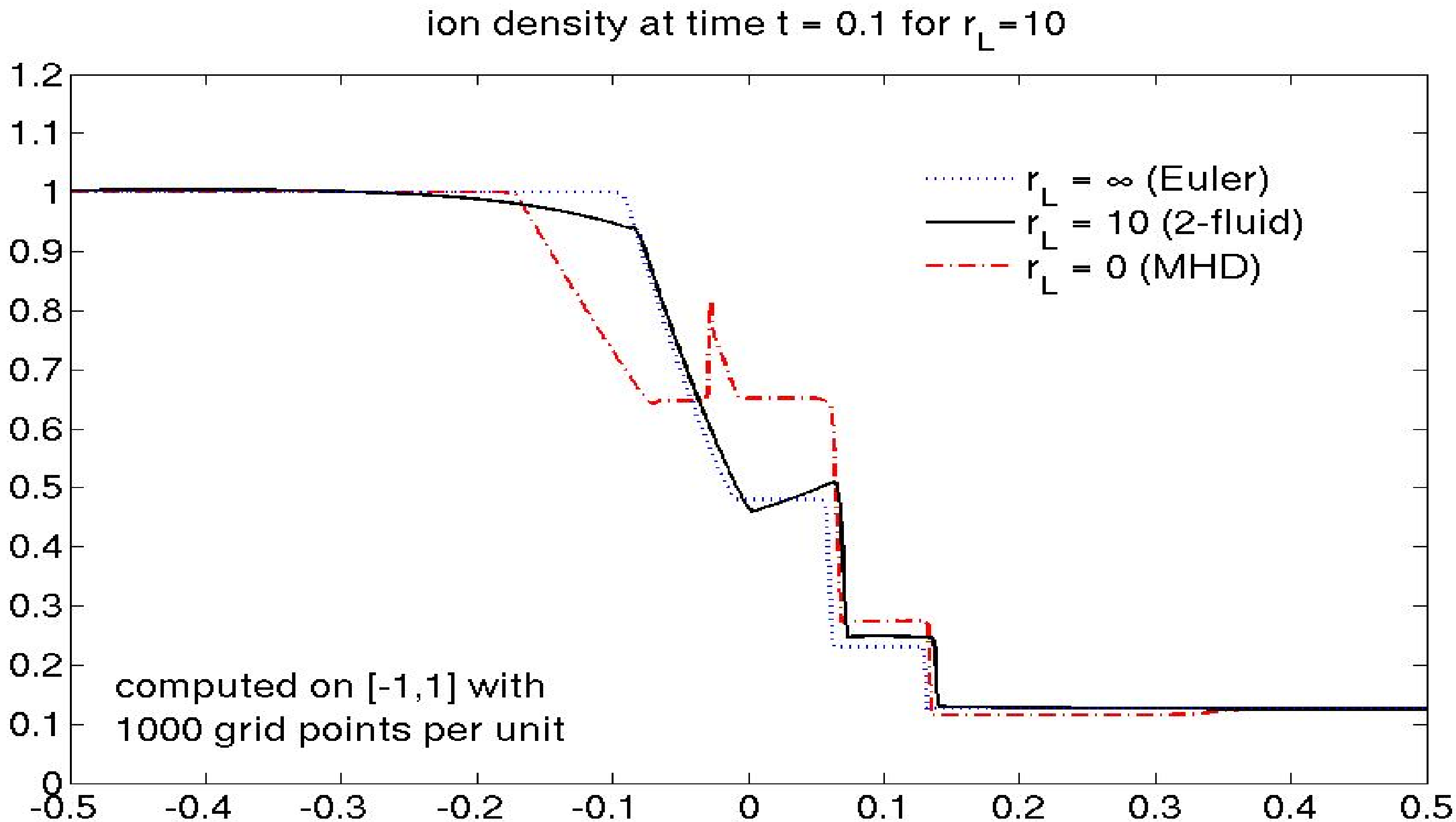
Numerical Method: Results.

We plotted ion density at nondimensionalized time $t = 0.1$ for a range of values of the nondimensionalized Larmor radius: $r_L = \infty$ (an Euler gas dynamics computation), $r_L = 10$, 1, 0.1, 0.01, 0.003, (two-fluid computations) and $r_L = 0$ (an ideal Magnetohydrodynamics computation).



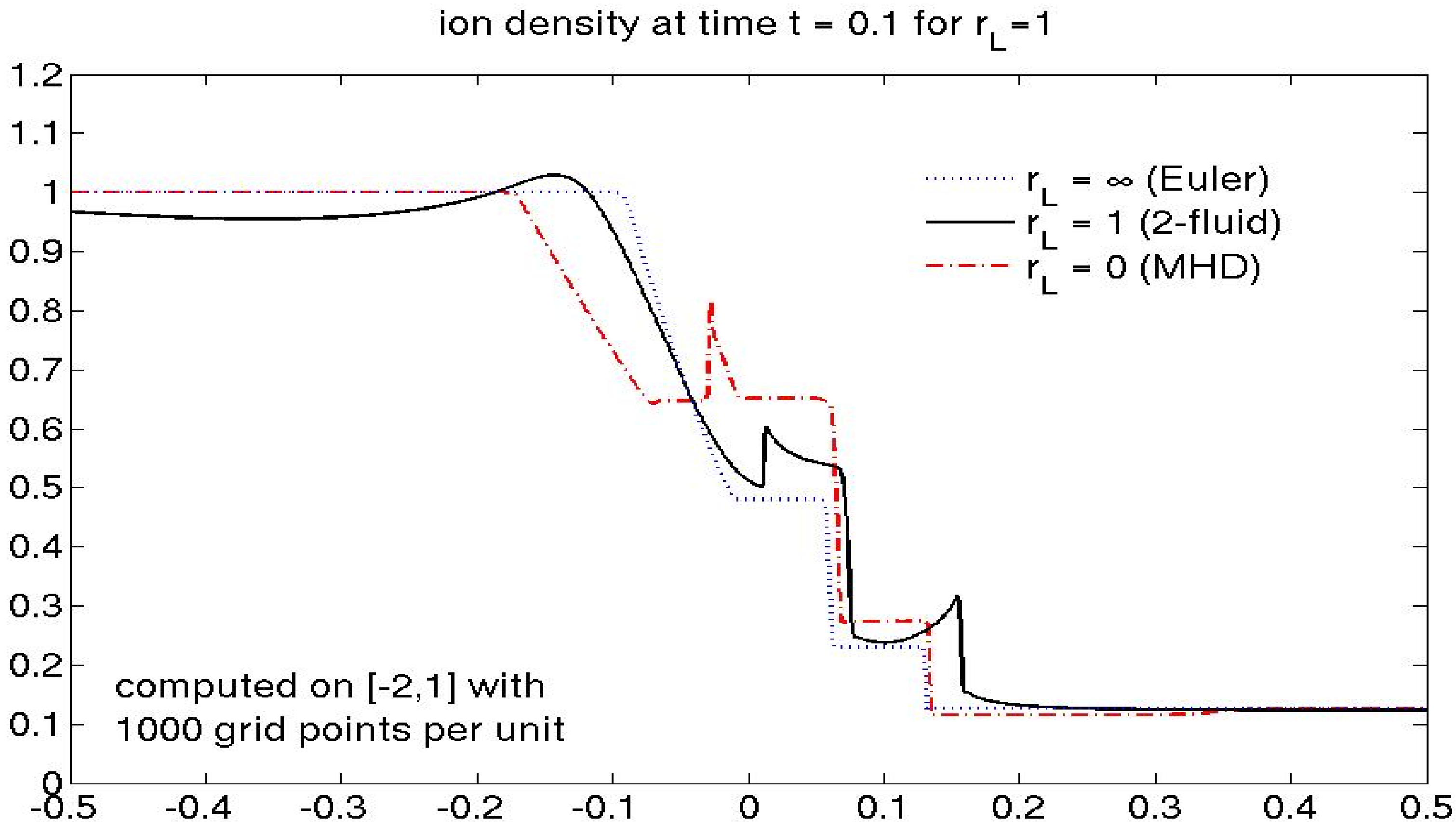
The initial ion density is piecewise constant with a single discontinuity at zero.

Numerical Method: Results.



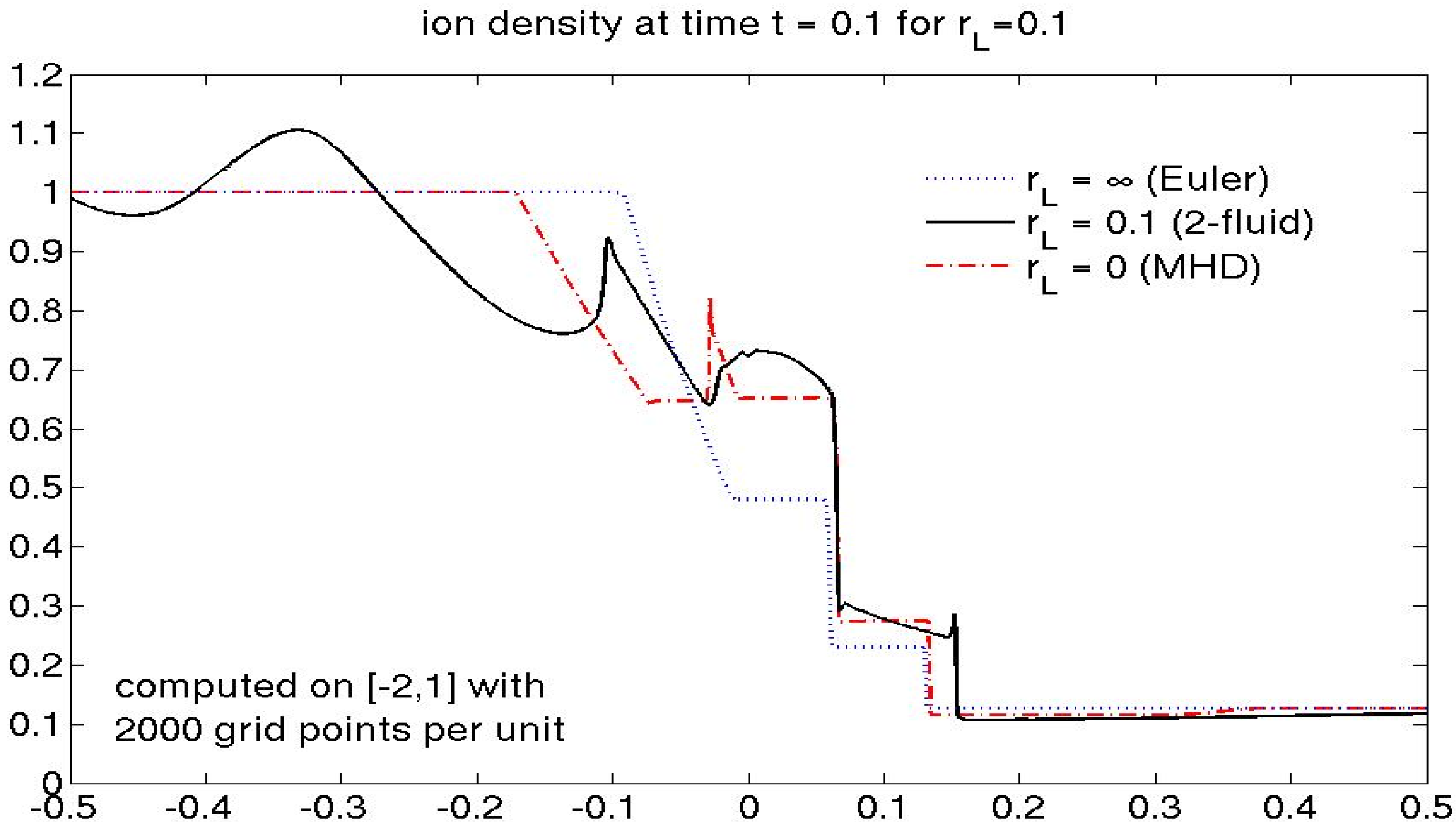
When the Larmor radius is large ($r_L = 10$), the electromagnetic effects are weak and the ions behave like an ideal gas. (At $r_L = 100$, 2-fluid is indistinguishable from Euler.)

Numerical Method: Results.



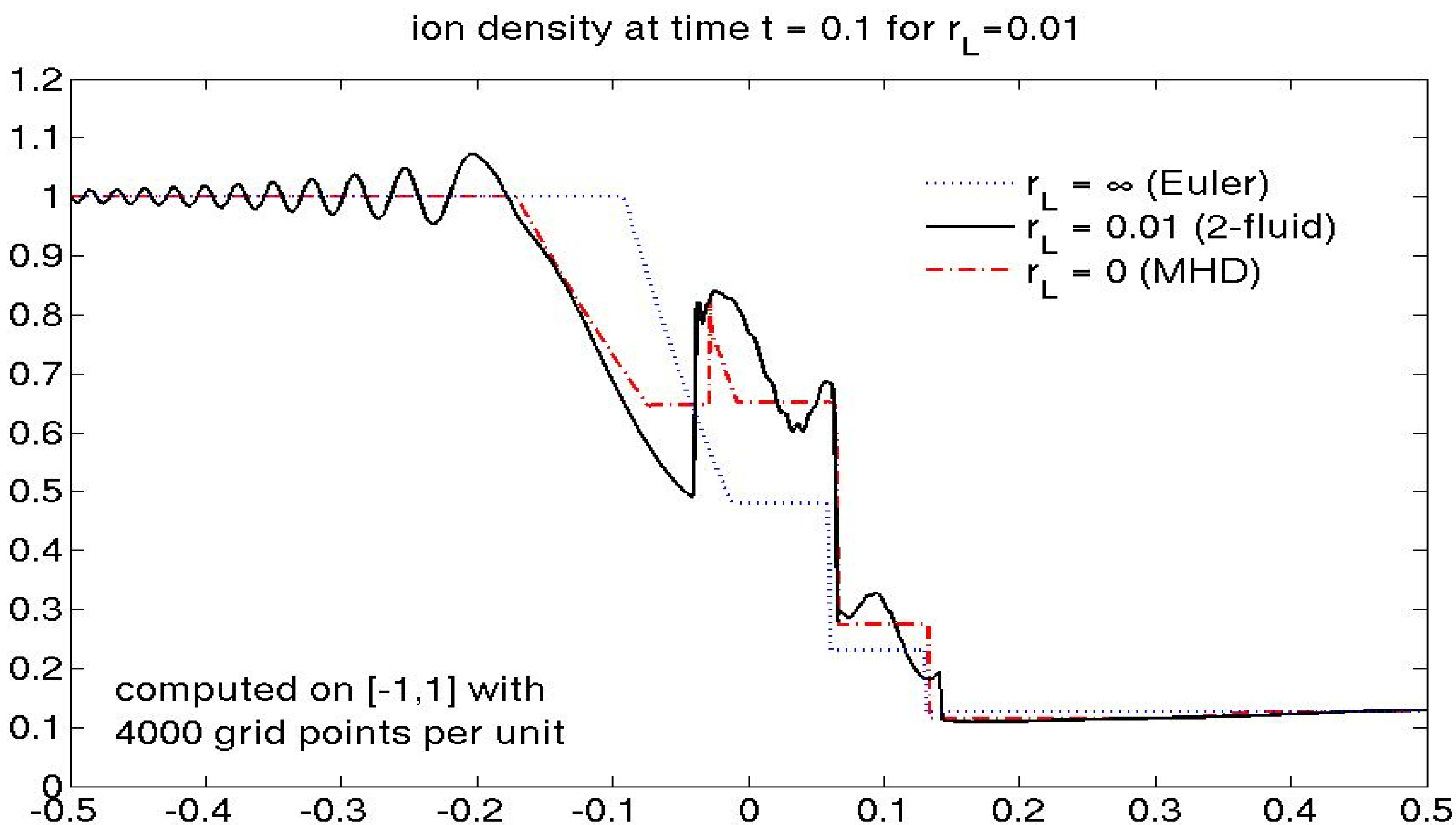
As we decrease the Larmor radius, the solution begins to transition away from gas dynamics (and eventually toward MHD).

Numerical Method: Results.



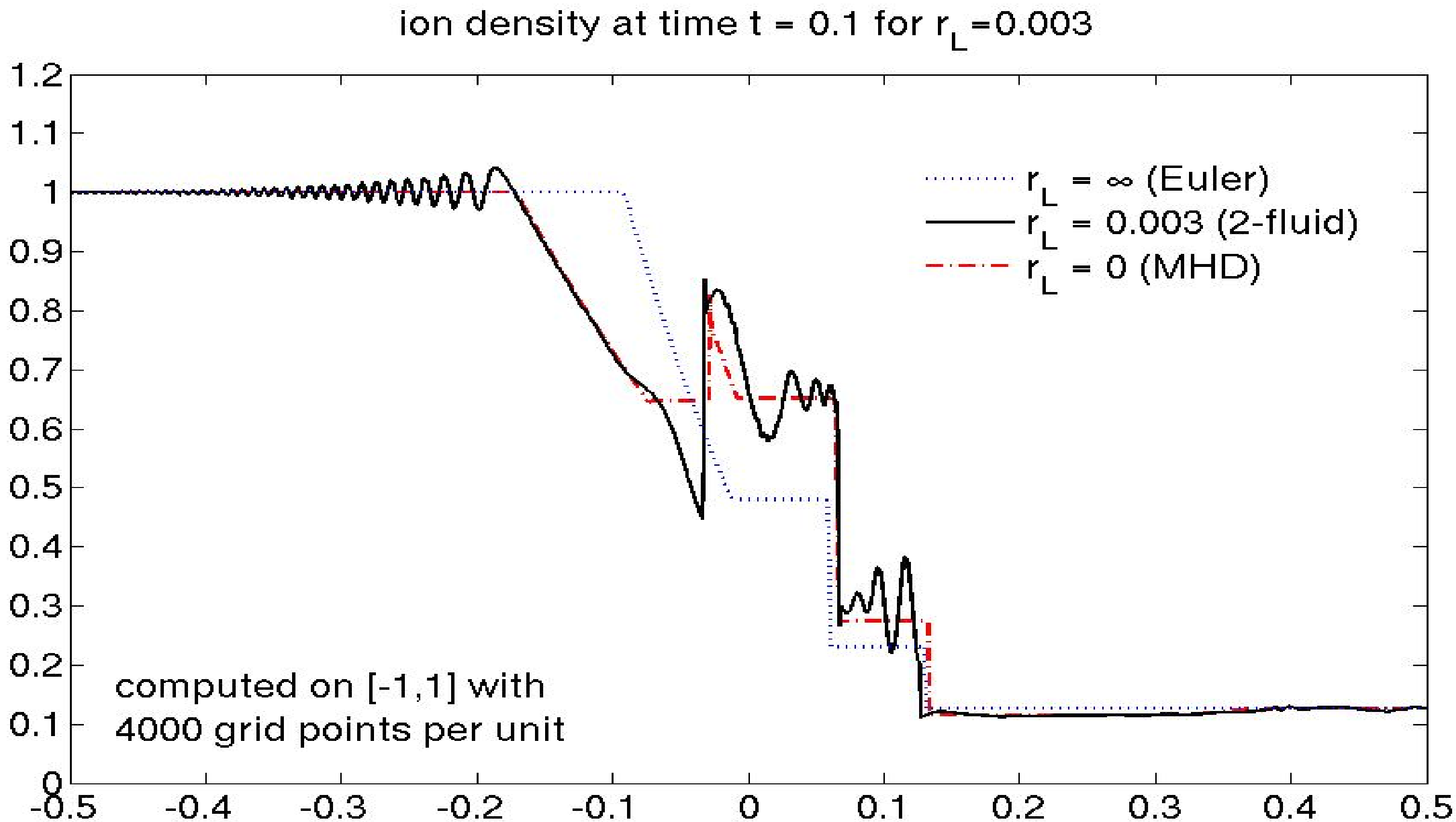
As the Larmor radius becomes smaller, higher-frequency oscillations begin to set in.

Numerical Method: Results.



As the Larmor radius becomes even smaller, the frequency of the oscillations increases and the solution begins to weakly approach the MHD solution.

Numerical Method: Results.



Convergence to MHD is suggested but far from confirmed. Unfortunately, computational expense increases with decreasing Larmor radius.

Future work.

① Accelerate solver by resolving fast waves and high frequencies only where needed (using techniques such as adaptive mesh refinement and implicit methods), and compare with Hall MHD.

② Extend solver to 2-dimensions.

③ Obtain a fast solution to the Geospace Environmental Modeling (GEM) reconnection challenge problem.

④ Extend solver to special and general relativistic flows.